On Hamiltonian Paths with Prescribed Edge Lengths in the Complete Graph

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Abstract

Marco Buratti has conjectured that given $p$ a prime and a multiset $S$ containing $p - 1$ non-zero elements from $\mathbb{Z}_p$, there exists a Hamiltonian path in $K_p$ where the multiset of edge lengths is $S$. In this paper we completely solve this conjecture when $S$ contains at most two distinct values.

1 Introduction

Given a graph $G$, a Hamiltonian Path in $G$ is a path that visits every vertex exactly once. In a given graph, a Hamiltonian path may or may not exist, however, it is well known in that complete graphs Hamiltonian paths always exist. Let $p$ be a prime and let $K_p$ be the complete graph on $p$ vertices. When the vertices of $K_p$ are labeled with the elements of the cyclic group $\mathbb{Z}_p$ one can define the length of an edge $xy$ to be $x - y$ or $y - x$, whichever is less than $\frac{p - 1}{2}$. Extending this definition slightly we say that an edge with length $k \leq \frac{p - 1}{2}$ also has length $p - k$ (there will be no ambiguity when this is used in context). The reader is referred to [4] for further definitions and graph theoretic background and to [1] for an extensive survey of Hamiltonian graphs. In 2007, Marco Buratti made the following conjecture.

Conjecture 1 (Buratti) Given $p$ a prime and a multiset $S$ containing $p-1$ non-zero elements from $\mathbb{Z}_p$, there exists a Hamiltonian path $H$ in $K_p$ where the multiset of edge lengths in $H$ is $S$.

From the above, it is clear that Buratti’s conjecture is equivalent to the following: Given $p$ a prime and a multiset $S$ containing $p-1$ non-zero
elements between 1 and \( \frac{p-1}{2} \), there exists a Hamiltonian path \( H \) in \( K_p \), where the multiset of edge lengths in \( H \) is \( S \). We first consider the two extremal cases, namely when \( |S| = 1 \) and when \( |S| = p - 1 \).

When \( |S| = 1 \) all edges have the same length \( k \). The path \( 0, k, 2k, 3k, \ldots, (p-1)k \) (with all terms reduced modulo \( p \)) is a Hamiltonian path with all edge lengths \( k \). Note that since \( k \) and \( p \) are relatively prime, all vertices in this path are distinct.

Now assume that \( |S| = p - 1 \) and hence \( S = \{1, 2, \ldots, p-1\} \). The following is a Hamiltonian path with the prescribed lengths: \( 0, 1, -1, 2, -2, 3, -3, \ldots, \frac{p-1}{2}, -\frac{p-1}{2} \) (again all terms are reduced modulo \( p \)). It is easy to check that the sequence of edge lengths starts with 1 and increases by 1 for each edge added, giving an edge of length \( d \) for all non-zero elements \( d \in \mathbb{Z}_p \). Note that the edge lengths of this Hamiltonian path can also be given as \( S = \{1, 1, 2, 2, \ldots, \frac{p-1}{2}, \frac{p-1}{2}\} \). This is essentially the well-known Walecki Construction (see [3] for a recent survey of this construction).

The main result of this paper is that when \( S \) consists of exactly two different values \( a \) and \( b \), that there is always a Hamiltonian path in \( K_p \) where the edges all have length either \( a \) or \( b \).

\section{Paths with Edge Lengths \( a \) and \( b \)}

The case of exactly two edge lengths is greatly simplified by the following two reductions.

**Lemma 2** Let \( p \) be a prime. The existence of a Hamiltonian path in \( K_p \) where all edges have length \( a \) or \( b \) is equivalent to the existence of a Hamiltonian path where all lengths are 1 and \( a^{-1}b \).

**Proof:** Since \( \mathbb{Z}_p \) is a field, multiplicative inverses exist for all elements of \( \mathbb{Z}_p \setminus \{0\} \). List the vertices of the Hamiltonian path with lengths \( a \) and \( b \) in the order in which they were visited, \( x_0, x_1, \ldots, x_{p-1} \). Since it is a Hamiltonian path, every vertex appears exactly once. Multiply each vertex by \( a^{-1} \). The resulting sequence is \( a^{-1}x_0, a^{-1}x_1, \ldots, a^{-1}x_{p-1} \). Since \( a^{-1} \) is a unit in \( \mathbb{Z}_p \), each element of \( \mathbb{Z}_p \) appears once in the new sequence, so it is still a Hamiltonian path. Furthermore, the lengths of each path are now either \( a^{-1} \ast a = 1 \) and \( a^{-1} \ast b \).

Clearly it doesn’t matter which length was chosen to be labeled \( a \) and which was chosen to be labeled \( b \). Hence, one can always label the length that occurs more often as \( a \) and the length that occurs less often as \( b \). This reduces Buratti’s entire problem with two edge lengths to showing that there is a Hamiltonian path with edge lengths 1 and \( k \) for all \( k \) where
number of occurrences of edges of length 1 is not less than the number occurrences of edges of length \( k \). We record this in the following proposition.

**Proposition 3** Let \( 1 < k \leq \frac{p-1}{2} \) and suppose that there exists a Hamiltonian path in \( K_p \) containing \( u \) edges of length 1 and \( v \) edges of length \( k \) for every \( u \) and \( v \) with \( u + v = p - 1 \) and \( u \geq v \). Then, given any \( a \) and \( b \) with \( 1 \leq a, b \leq p - 1 \) and any \( n_a \) and \( n_b \) with \( n_a + n_b = p - 1 \) there exists a Hamiltonian path \( H \) in \( K_p \) with exactly \( n_a \) edges of length \( a \) and \( n_b \) edges of length \( b \).

### 3 Constructions

For the remainder of the paper we will be constructing Hamiltonian paths in \( K_p \) \((p \geq 5 \text{ a prime})\) with all edges of length 1 or \( k \) with \( 1 < k \leq \frac{p-1}{2} \). Let \( n \) denote the number of edges of length \( k \) in the Hamiltonian path and hence there are \( p - 1 - n \) edges of length 1 in the path. From Proposition 3 we can assume that \( 1 \leq n \leq \frac{p-1}{2} \). Using the division algorithm, we write \( n = qk + r \) where \( 0 \leq r < k \). We require four fairly similar constructions to cover all the cases. We consider the cases when \( k \) is even and when \( k \) is odd with \( r \) even, \( r \) odd, and \( q = 1 \).

#### 3.1 \( k \) even

**Lemma 4** If \( n = qk + r \) with \( k \) even, then there is a Hamiltonian path in \( K_p \) with \( n \) edges of length \( k \) and \((p - 1) - n \) edges of length 1.

**Proof:** First consider the case where \( n = qk + r \) with \( k \) even, \( r \) even. We begin by placing all the edges of length \( k \) and construct a path in the following manner:

Start at 0 and go upwards in increments of \( k \) until the vertex \((q + 1)k \) is reached. This picks up all of the vertices congruent to 0 modulo \( k \) between 0 and \((q + 1)k \). Add one to get to \((q + 1)k + 1\), and then travel backwards in increments of \( k \). This picks up all of the vertices congruent to 1 modulo \( k \). At 1, add one to get to the vertex 2 and then travel to the vertices congruent to 2 modulo \( k \) by way of paths of length \( k \). Continue in this pattern until you have reached the vertex \( r - 1 \). At this point, add 1 to get to the vertex \( r \) and travel upwards in increments of \( k \). Instead of going to \((q + 1)k + r \), end at \( qk + r \). Add 1 and travel downwards to \( r + 1 \). Pick up the remaining congruence classes (modulo \( k \)) in this manner. Since \( k \) is even, the last congruence class will be obtained by traveling downward and ending at \( k - 1 \). This part of the Hamiltonian path containing the edges of length \( k \) is summarized in the list below:
\[0, k, 2k, \ldots, (q + 1)k,\]
\[(q + 1)k + 1, qk + 1, \ldots, 1,\]
\[2, k + 2, 2k + 2, \ldots, (q + 1)k + 2,\]
\[(q + 1)k + 3, qk + 3, \ldots, 3,\]
\[\cdots\]
\[r - 2, k + (r - 2), \ldots, (q + 1)k + (r - 2),\]
\[(q + 1)k + (r - 1), qk + (r - 1), \ldots, r - 1,\]
\[r, k + r, 2k + r, \ldots, qk + r,\]
\[qk + r + 1, \ldots, r + 1,\]
\[r + 2, \ldots, qk + (r + 2),\]
\[\cdots\]
\[qk + (k - 1), \ldots, k - 1\]

Note that if \( r = 0 \), it is necessary to start at the row beginning with \( r \) and not the row listed above as beginning with 0.

This path exhibits certain properties. First, it is easy to see that every vertex between 0 and \((q + 1)k + (r - 1)\) has been visited. Furthermore, the degree of every vertex in that range is 2 with the exceptions of 0 and \( k - 1 \). Second, the path contains exactly \( n \) edges of length \( k \). This follows since the first \( r \) congruence classes have \( q + 1 \) edges of length \( k \) going forward and the last \( k - r \) congruence classes contain exactly \( q \) edges going forward giving a total of \( r(q + 1) + (k - r)q = qk + r = n \) edges of length \( k \). Lastly, the vertex 0 has an open end, so the additional edges of length 1 can be added in a counterclockwise manner starting at 0 until reaching the vertex \((q + 1)k + r\). The addition of these edges of length 1 yields a Hamiltonian path.

In the case where \( k \) is even and \( r \) is odd, the same construction above can be used with one slight modification. Let \( r = s + 1 \) and use \( s \) in the construction. The path contains \( qk + r - 1 \) edges of length \( k \) and ends at \( k - 1 \). Add the edge \((k - 1, -1)\). This time the open vertex is at \(-1\), so the extra edges of length 1 needed to make a Hamiltonian path are added this time in a counterclockwise manner starting at \(-1\).

The last property that needs to be checked is that this construction fits in \( K_p \) for all \( p \). In the case of \( r \) even, the largest vertex visited is \((q + 1)k + r - 1\). In the case of \( r \) being odd, one more vertex was used, so if the vertices are relabeled so the open end is at 0, the largest vertex visited is \((q + 1)k + r\) (we must show this value is less than \( p \)). Recall that by Proposition 3, \( n = qk + r \leq \frac{p - 1}{2} \) and \( k \leq \frac{p - 1}{2} \). It follows that, \((q + 1)k + r = (qk + r) + k \leq \frac{p - 1}{2} + \frac{p - 1}{2} = p - 1 \). Therefore, this construction works for all \( p \).
Figures 1 and 2 show an example of the construction when \( k \) is even. Figure 1 shows a path where \( k \) is even and \( r \) is even. In this example, 10 edges of length 4 are placed in \( \mathbb{Z}_{23} \). In this case, \( n = qk + r = 2 \times 4 + 2 \). Figure 2 shows how the case where \( k \) is even and \( r \) is even can be extended to \( r \) being odd by adding the edge \((k - 1, -1)\). For this case we demonstrate 11 edges of length 4 in \( \mathbb{Z}_{23} \).

![Figure 1: A Hamiltonian path with \( p = 23, n = 10, k = 4 \)](image1)

![Figure 2: A Hamiltonian path with \( p = 23, n = 11, k = 4 \)](image2)

### 3.2 \( k \) odd

It is necessary to break this case into three parts. Lemma 5 covers the case when \( k \) is odd and \( r \) is even. Lemmas 6 and 7 cover the case when \( k \) is odd and \( r \) is odd with Lemma 6 providing a construction for \( q \geq 2 \) and Lemma 7 providing the construction for \( q = 1 \).
Lemma 5 If \( n = qk + r \) with \( k \) odd and \( r \) even, then there is a Hamiltonian path in \( K_p \) with \( n \) edges of length \( k \) and \( (p - 1) - n \) edges of length 1.

Proof: In the case of \( k \) odd and \( r \) even, we use essentially the same pattern that was used in the \( k \) even case with even remainder. However note that instead of ending the path at the vertex \( k - 1 \), the path now ends at \( qk + k - 1 \). This change is noted in the summary of the pattern below.

\[
0, k, 2k, \ldots, (q + 1)k, \\
(q + 1)k + 1, qk + 1, \ldots, 1, \\
2, k + 2, 2k + 2, \ldots, (q + 1)k + 2, \\
(q + 1)k + 3, qk + 3, \ldots, 3, \\
\ldots \\
r - 2, k + (r - 2), \ldots, (q + 1)k + (r - 2), \\
(q + 1)k + (r - 1), qk + (r - 1), \ldots, r - 1, \\
r, k + r, 2k + r, \ldots, qk + r, \\
qk + r + 1, \ldots, r + 1, \\
r + 2, \ldots, qk + (r + 2), \\
\ldots \\
k - 1, \ldots, qk + (k - 1)
\]

Once again, this path visits the first \((q + 1)k + (r - 1)\) vertices and contains exactly \( n = qk + r \) edges of length \( k \), since again the first \( r \) congruence classes have \( q + 1 \) edges of length \( k \) going forward and the last \( k - r \) congruence classes contain exactly \( q \) edges going forward. To extend this path to a Hamiltonian path, the remaining edges of length 1 are be added in a counterclockwise manner starting at the open end, i.e. at the vertex 0.

Finally, it is necessary to check that this construction will fit into \( K_p \). Once again, the highest vertex visited is \((q + 1)k + r - 1\). The conditions that \( n = qk + r \leq \frac{p - 1}{2} \) and \( k \leq \frac{p - 1}{2} \) still hold by Proposition 3. It follows that \((q + 1)k + r - 1 = (qk + r) + k - 1 \leq \frac{p - 1}{2} + \frac{q - 1}{2} = p - 2\), completing the proof.

Figure 3 shows the construction in Lemma 5 for \( p = 29, k = 5, \) and \( n = 12 = 2 \ast 5 + 2 \).

Lemma 6 If \( n = qk + r \) with \( k \) odd, \( r \) odd and \( q \geq 2 \), then a Hamiltonian path with \( n \) edges of length \( k \) and \( (p - 1) - n \) edges of length 1 exists.

Proof: Similar to what was done with the case where \( k \) is even, the construction for \( k \) odd and \( r \) even can be extended to \( k \) odd and \( r \) odd by adding one edge. As before, we do the construction for \( k \) odd and \( r \) even.
using $s = r - 1$ as the remainder. This time the path ends at the vertex $qk + k - 1$. Add in the edge $(qk + k - 1, qk + 2k - 1)$. The vertices between $(q + 1)k + r - 1$ and $(q + 2)k - 1$ can be reached by paths of length 1 by starting at the open end of the path at $qk + 2k - 1$ and subtracting 1. The vertices between $qk + 2k - 1$ and 0 can be reached by paths of length 1 where 0 is the starting point and the edges proceed counterclockwise.

In this case, it is necessary to show that $qk + 2k - 1 \leq p - 1$. By Proposition 3, $qk + r \leq \frac{p - 1}{2}$. Since $qk \leq qk + r$, it follows that $qk \leq \frac{p - 1}{2}$. Now using the hypothesis that $q \geq 2$ we get that $qk + 2k - 1 \leq qk + qk - 1 \leq \frac{p - 1}{2} + \frac{p - 1}{2} - 1 \leq p - 1$, completing the proof.

Figure 4 shows the construction from Lemma 6 for $p = 29$, $n = 13$, and $k = 5$. Note that the constructed path is the same as in Figure 3 with the addition of the edge $(14, 19)$ and the subtraction of the edge $(19, 20)$.
In the case where \( q = 1 \), Lemma 6 will not work when \( 2k \) is greater than \( \frac{p-1}{2} \). Our final lemma gives a construction for the \( q = 1 \) case.

**Lemma 7** If \( n = qk + r \) with \( k \) odd, \( r \) odd and \( q = 1 \), then a Hamiltonian path with \( n \) edges of length \( k \) and \( (p - 1) - n \) edges of length 1 exists.

**Proof:** Consider the following construction: Place an edge of length \( k \) from 0 to \( k \). Instead of going forward 1 (as in the previous constructions), subtract 1 to get to the vertex \( k - 1 \). At this point go forward \( k \) to the vertex \( 2k - 1 \). Add in two edges of length 1 by going forwards to \( 2k \) and \( 2k + 1 \). From here, the pattern is similar to previous constructions. Travel backwards and pick up the class of numbers congruent to 1 modulo \( k \). Add 1 and travel forwards by lengths of \( k \), which adds the vertices congruent to 2 modulo \( k \) to the path. Continue in this pattern until the vertex \( r \) is reached by traveling backwards from \( 2k + r \). This gives \( 2k + r \) as the largest vertex visited thus far. At this point use the following pattern: add 1, add \( k \), add 1, subtract \( k \). This continues until the vertex \( k - 2 \) is reached by traveling backwards from \( 2k - 2 \). This path is summarized in the following list:

\[
0, k, \\
k - 1, 2k - 1, 2k, \\
2k + 1, k + 1, 1, \\
2, k + 2, 2k + 2, \\
\ldots \\
r - 1, k + r - 1, 2k + r - 1, \\
2k + r, k + r, r, \\
r + 1, k + r + 1, \\
k + r + 2, r + 2, \\
\ldots \\
r - 3, k + (r - 3)k + (r - 2), r - 2
\]

This path indeed contains the correct number of edges of length \( k \) since every vertex between 0 and \( k + r \) has an edge of length \( k \) traveling forward with the exception of vertex \( k \). Also, it is easy to see that there are no isolated vertices between 0 and \( k + r \). Lastly, the end at 0 is open, which enables us to pick up the vertices between \( 2k + r \) and 0 with edges of length 1.

The last thing to check is that the construction fits in \( K_p \). By Proposition 3, \( n = k + r \leq \frac{p-1}{2} \). Also \( k \leq \frac{p-1}{2} \). Therefore, \( 2k + r = k + k + r \leq \frac{p-1}{2} + \frac{p-1}{2} = p - 1 \), completing the proof. \( \blacksquare \)

Figure 5 shows the construction from Lemma 7 for \( p = 23, k = 7 \), and \( n = 10 = 1 \ast 7 + 3 \).
4 Conclusion

We combine the constructions of the previous section to get the following theorem.

**Theorem 8** Let $p$ a prime, $k \leq \frac{p-1}{2}$, and $n \leq \frac{p-1}{2}$, then there exists a Hamiltonian path in $K_p$ containing $n$ edges of length $k$ and $(p-1) - n$ edges of length $1$.

**Proof:** This theorem follows directly from Lemmas 4, 5, 6, and 7.

From the reductions of Section 2 and Theorem 8 we now have a solution to Buratti’s Conjecture for the case of two distinct edge lengths.

**Theorem 9** Given $p$ a prime, $1 \leq n \leq p-1$ and any nonzero lengths $a, b$ with $\{a, b\} \in \mathbb{Z}_p$, there exists a Hamiltonian path in $K_p$ containing $n$ edges of length $a$ and $(p-1) - n$ edges of length $b$.

**Proof:** By Theorem 8, there exists a Hamiltonian path in $\mathbb{Z}_p$ containing $n$ edges of length $k$ and $(p-1) - n$ edges of length 1. The existence of this path in conjunction with Proposition 3 give the existence of the desired path.

We note here that Theorem 4 was proven independently by Horak and Rosa in [2]. The interested reader is referred to that paper for further discussion of Buratti’s problem.

The following corollary deals with the case of Hamiltonian paths in complete graphs of nonprime order.

**Corollary 10** Given $s \in \mathbb{N}$ and $n \leq \frac{s-1}{2}$, there exists a Hamiltonian path in $K_s$ containing $n$ edges of length $m$ and $(s-1) - n$ edges of length 1.
Proof: Examining the proofs of Lemmas 4, 5, 6, and 7, it can be observed that the condition of \( s \) being a prime was not used. Therefore, the constructions work for any \( s \in \mathbb{N} \).

Our final corollary deals with Hamiltonian paths with edges of length \( a \) and \( b \) in \( K_s \) with \( s \) not prime.

**Corollary 11** Given \( s \in \mathbb{N} \) and assume that \( a \) and \( b \) are nonzero lengths in \( \mathbb{Z}_s \) with \( \text{gcd}(s, a) = 1 \) and \( \text{gcd}(s, b) = 1 \), then there exists a Hamiltonian path in \( K_s \) containing \( n \) edges of length \( a \) and \( (s - 1) - n \) edges of length \( b \) for all \( 0 \leq n \leq s - 1 \).

**Proof:** The case of \( n = 0 \) or \( n = p - 1 \) is solved by considering the path consisting of the successive multiples of \( a \) (or \( b \)) in the cyclic group \( \mathbb{Z}_s \). Now without loss of generality assume that \( n \leq \frac{s - 1}{2} \). Since \( b \) is a unit in \( \mathbb{Z}_s \), the existence of a Hamiltonian path in \( K_s \) with \( n \) edges of length \( a \) and \( (s - 1) - n \) edges of length \( b \) is implied by the existence of a Hamiltonian path in \( K_s \) with \( n \) edges of length \( ab^{-1} \) and \( (s - 1) - n \) edges of length 1. The result now follows from Corollary 10.

A basic computer search conducted using *Mathematica* has shown Bu-ratti’s Conjecture is true for \( p = 7 \). The following is a table that contains every possible multiset \( S \) of lengths in \( \mathbb{Z}_7 \) and a Hamiltonian path in \( K_7 \) whose edge lengths correspond to the values in the multi-set \( S \).

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The cases in \( K_8 \) and \( K_9 \) were also run on the computer. In the case of \( K_8 \), there are 120 possible multi-sets \( S \) and 105 of them are realizable.
as Hamiltonian paths. In the case of $K_9$, there are 165 multi-sets $S$ with 161 of these having realizable Hamiltonian paths. The only four sets that do not have realizable paths are \{1, 3, 3, 3, 3, 3, 3\}, \{2, 3, 3, 3, 3, 3, 3\}, \{3, 3, 3, 3, 3, 3, 3\} and \{3, 3, 3, 3, 3, 3, 4\}. Clearly these paths are not realizable as it is not possible to attain elements of more than 2 of the congruence classes modulo 3. It is interesting to note that these are the only multisets in $\mathbb{Z}_9$ with no realizable path.

References


