BOSS BLOCK DESIGNS

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Abstract

A well-known inequality of Bose states that in a BIBD containing a parallel class of blocks $b \geq v + r - 1$. We consider BIBDs which contain $m$ parallel classes, each containing a fixed block but otherwise disjoint. We obtain $b \geq v + r - 1 + r \frac{(m-1)(k-1)}{r-m}$ BIBDs in which equality holds, which we refer to as boss block designs, are investigated in the remainder of the paper.

1. Introduction.

A balanced incomplete block design, or BIBD, with parameters $(v, b, r, k, \lambda)$ is a pair $(X, \beta)$, where $X$ is a set of size $v$, $\beta$ is a family of $b$ $k$-subsets of $X$ such that each element of $X$ occurs in $r$ members of $\beta$, and each pair of elements of $X$ occurs in $\lambda$ members of $\beta$. Henceforth, we refer to elements of $X$ as varieties and members of $\beta$ as blocks. It is immediate that $vr = bk$ and $\lambda(v-1) = r(k-1)$. A set $P$ of blocks of $\beta$ is a parallel class if each variety of $X$ occurs in exactly one block of $P$. A BIBD is said to be resolvable if $\beta$ can be partitioned into parallel classes. In [1], Bose proves that the parameters of a resolvable BIBD satisfy the inequality $b \geq v + r - 1$.

However, as indicated by Stanton and Sprott [6], this inequality is true for any BIBD containing a parallel class. In this paper we generalize the above inequality by supposing a BIBD contains $m$ parallel classes each of which contains a fixed block, $B$, but otherwise disjoint.

In the next section, we derive an inequality for BIBDs with such a block $B$. In the case $m = 1$, our inequality reduces to Bose's inequality stated above.

In the following sections, we also consider designs in which our inequality is satisfied as an equality and give some examples.

2. An inequality.

The following is our main result.

THEOREM 2.1. Suppose a BIBD $(X, \beta)$ contains a block $B$ which occurs in $m$ parallel classes. Also, suppose no two of these parallel classes intersect

in any block other than B. Then \( b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r-m} \). Further, if \( b = v + r - 1 + \frac{r(m-1)(k-1)}{r-m} \) then B intersects any other block in \( 0 \) or \( \mu \) varieties, where \( \mu = \frac{k^2(r-1)}{bk-mv+mk-k} \).

**Proof.** There are \( b-m(\frac{\nu}{k}-1)-1 \) blocks other than the blocks in the \( m \) parallel classes. Denote these blocks \( B_1, B_2, \ldots, B_k \), where \( \lambda = b-m(\frac{\nu}{k}-1)-1 \). Let \( b_i = |B \cap B_i| \) for \( 1 \leq i \leq \lambda \). Then we have

\[
\sum_{i=1}^{\lambda} b_i = k(r-1)
\]

and

\[
\sum_{i=1}^{\lambda} \binom{b_i}{2} = \binom{k}{2} (\lambda - 1).
\]

Thus

\[
\sum_{i=1}^{\lambda} b_i^2 = k(k\lambda - k - \lambda + r).
\]

Denote

\[
\mu = \sum_{i=1}^{\lambda} \frac{b_i}{\lambda} = \frac{k^2(r-1)}{bk-mv+mk-k}.
\]

Then

\[
\sum_{i=1}^{\lambda} (b_i - \mu)^2 = k(k\lambda - k - \lambda + r) - \frac{k^3(r-1)^2}{bk-m(v-k)-k}.
\]

Clearly \( 0 \leq \sum_{i=1}^{\lambda} (b_i - \mu)^2 \). Using \( \lambda(v-1) = r(k-1) \) and \( vr = bk \), and simplifying, we obtain

\[
0 \leq (v-k)[br - r^2 - r(v-k)-m(rk - 2r - v + 1 + b)].
\]

Thus

\[
b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r-m}.
\]

Also, \( b = v + r - 1 + \frac{r(m-1)(k-1)}{r-m} \) if and only if \( b_i = \mu \) for every \( 1 \leq i \leq \lambda \). □

If \( m = 1 \), then we obtain Bose's result \( b \geq v + r - 1 \). A resolvable BIBD has \( b = v + r - 1 \) if and only if any two blocks from different
parallel classes intersect in \( m \) varieties. Such BIBDs are called affine resolvable and have been extensively studied. See for example, [4,5].

A comment regarding Theorem 2.1 is in order. That is the assumption that there exist \( m\left(\frac{v}{k} - 1\right) \) blocks, other than \( B \), in \( m \) parallel classes may be weakened. It is necessary only to assume that these blocks be disjoint from \( B \), and contain each variety not in \( B \) \( m \) times. However, our main interest is in designs containing the \( m \) parallel classes as described above.


We now consider BIBDs which satisfy the above inequality as an equality. Thus, suppose a BIBD contains \( m \) parallel classes, each containing a fixed block \( B \) but otherwise disjoint. Further, suppose \( b = v + r - 1 + \frac{(m-1)r(k-1)}{r - m} \).

We refer to such a BIBD as an \((m, \mu)\) boss block design, henceforth BBIBD, and \( B \) as a boss block. \( B \) intersects each block not in the \( m \) parallel classes in \( \mu \) varieties, where \( \mu = \frac{k^2(r-1)}{bk-mv+mk-k} \). Thus, for example, an affine resolvable BIBD is a \((1, k^2/v)\) BBIBD (see Bose [1]).

We remark that if a BIBD is an \((m, \mu)\) BBIBD with boss block \( B \), and also an \((m', \mu')\) BBIBD with boss block \( B' \), then \( m = m' \) and \( \mu = \mu' \).

This is true since \( b = v + r - 1 + \frac{(m-1)r(k-1)}{r - m} \) determines \( m \) uniquely, and then \( \mu = \frac{k^2(r-1)}{bk-mv+mk-k} \) determines \( \mu \) uniquely. Thus there is no possibility of ambiguity when we speak of an \((m, \mu)\) BBIBD.

We now derive a few lemmata relating the various parameters of a BBIBD.

The following equality is an immediate consequence of the definition of BBIBD.

**Lemma 3.1.** In an \((m, \mu)\) BBIBD, \( b = v + r - 1 + \frac{r(m-1)(k-1)}{r - m} \).

**Lemma 3.2.** In an \((m, \mu)\) BBIBD, \( r = m + \frac{k\lambda}{\mu} \).

**Proof.** Let \( v \) be a variety not in the boss block. Counting ordered pairs \((C, w)\), with \( w \) in the boss block and \( \{v, w\} \subseteq C \), we obtain \( \mu(r-m) = k\lambda \).

**Lemma 3.3.** In an \((m, \mu)\) BBIBD, \( \lambda = 1 + \frac{(m-1)(r-1)}{k-1} \).

**Proof.** Let \( v \) be a variety not in the boss block. Counting ordered pairs \((C, w)\), with \( w \) in the boss block and \( \{v, w\} \subseteq C \), \( C \neq B \), we obtain \((\lambda-1)(k-1) = (\mu-1)(r-1) \). \( \square \)
LEMMA 3.4. In an \((m, \mu)\) BBIBD, \(v = \frac{(k-1)^2 + (\mu-1) \cdot r + (k-\mu)}{(\mu-1)r + (k-\mu)}\).

**Proof.** In any BIBD, \(v = \frac{r(k-1) + \lambda}{\lambda}\).

The result follows from Lemma 3.3.

LEMMA 3.5. In an \((m, \mu)\) BBIBD, \(m = \frac{(r-k)(k-\mu)}{\mu(k-1)}\).

**Proof.** The result follows from Lemma 3.2 and Lemma 3.3.

We now derive inequalities relating \(\mu\) to \(k\).

LEMMA 3.6. In an \((m, \mu)\) BBIBD, \(\mu \leq k/2\).

**Proof.** Suppose first that \(\mu \geq k/2 + 1\).

Then, from Lemma 3.4, \(v = \frac{(k-1)^2 + (\mu-1) \cdot r + (k-\mu)}{(\mu-1)r + (k-\mu)}\)

\[
< \frac{(k-2)r + k/2}{r(k-2)r} = \frac{2(k-1)r + 1}{r} < 2(k-1) + 1 = 2k-1.
\]

However, if a BIBD contains a parallel class \(k|v\), so \(v \geq 2k\). Thus we have a contradiction.

If \(\mu = \frac{k+1}{2}\), then \(v = \frac{(k-1)^2 + (\frac{k-1}{2})r + \frac{k-1}{2}}{(\frac{k-1}{2})r + (\frac{k-1}{2})} = \frac{(2k-1)r + 1}{r + 1} < 2k\), again yielding a contradiction.

LEMMA 3.7. An \((m, \mu)\) BBIBD with \(\mu = \frac{k}{2}\ and \ \mu > 1\ has \ v = 2k\ and the parameters of an affine resolvable BIBD (i.e., \(m = 1\)).

**Proof.** From Lemma 3.4,

\[
v = \frac{(k-1)^2 + \frac{k}{2} - 1) \cdot r + \frac{k}{2}}{(\frac{k}{2} - 1)r + \frac{k}{2}} = \frac{(2k^2 - 4k)r + rk + k}{(k - 2)r + k}
\]

\[
< \frac{(2k^2 - 4k)r + rk + k}{(k-2)r} + \frac{rk + k}{rk + k-2r} = 2k + 1 + \frac{2r}{rk + k-2r}
\]

\[
< 2k + 1 + \frac{2r}{4r + 4 - 2r} = 2k + 1 + \frac{2r}{2r + 4} < 2k + 2
\]

Thus \(v < 2k + 1\). Since \(k|v\) therefore \(v = 2k\). Now, since \(\mu = \frac{k}{2}\), it is clear that \(m = 1\). (Note that \(k \geq 4\) by the assumptions.)

Finally, we determine the parameters of an \((m,1)\)-BBIBD.
LEMMA 3.8. An \((m, \mu)\) BBIBD with \(\mu = 1\) has parameters \((t(k^2 - k) + k, (tk + 1)(k - 1) t + 1), tk + 1, k, 1)\) for a positive integer \(t\). Also, \(m = r - k\).

Proof. By Lemma 3.3, \(\lambda = 1\). Thus \(v = r(k - 1) + 1\). Since \(k|v\), thus \(r - 1 = tk\) for some \(t\). Since \(b = \frac{vr}{k}\), the parameters have the desired form. By Lemma 3.5, \(m = r - k\).

4. BBIBDs with \(k \leq 7\).

In this section, we determine the parameters of possible BBIBDs with \(k \leq 7\).

THEOREM 4.1. A BBIBD with \(k = 2\) is a \((2t + 2, \binom{2t + 2}{2}, 2t + 1, 2, 1)\) BIBD. Such a BIBD is a \((2t - 1, 1)\) BBIBD, and exists for any positive integer \(t\).

Proof. By Lemma 3.6, \(\mu = 1\), if \(k = 2\). By Lemma 3.8, the parameters are as stated above. We show the desired designs exist. Given a \((2t + 2)\) set \(V\), take \(\beta\) as all 2-subsets of \(V\). Choose any block \(B\) as the boss block.

THEOREM 4.2. A BBIBD with \(k = 3\) is a \((6t + 3, (2t + 1)(3t + 1), 3t + 1, 3, 1)\) design, with \(m = 3t + 2, \mu = 1\).

Proof. This is a direct result of Lemma 3.6 and Lemma 3.8.

EXAMPLE 4.1. There exists a \((15, 35, 7, 3, 1)\) BIBD which is a \((4, 1)\) BBIBD.

0 5 10 is the boss block.

\[
\begin{array}{cccccccccccc}
0 & 5 & 10 \\
1 & 6 & 11 & 2 & 3 & 6 & 3 & 4 & 7 & 1 & 3 & 9 \\
2 & 7 & 12 & 4 & 11 & 13 & 12 & 13 & 1 & 4 & 6 & 12 \\
3 & 8 & 13 & 7 & 14 & 1 & 6 & 8 & 14 & 7 & 8 & 11 \\
4 & 9 & 14 & 8 & 9 & 12 & 9 & 11 & 2 & 13 & 14 & 2 \\
1 & 2 & 5 & 2 & 4 & 10 & 10 & 11 & 14 & 10 & 12 & 3 \\
4 & 5 & 8 & 3 & 5 & 11 & 11 & 12 & 0 & 12 & 14 & 5 \\
5 & 6 & 9 & 5 & 7 & 13 & 14 & 0 & 3 & 13 & 0 & 6 \\
6 & 7 & 10 & 7 & 9 & 0 & 0 & 1 & 4 & 0 & 2 & 8 \\
9 & 10 & 13 & 8 & 10 & 1 \\
\end{array}
\]
THEOREM 4.3. A BBIBD with \( k = 4 \) is a \((12t + 4, (3t + 1)(4t + 1), 4t + 1, 4, 1)\) BBIBD with \( m = 4t - 3, \mu = 1 \), or an \((8, 14, 7, 4, 3)\) BBIBD with \( m = 1 \).

**Proof.** If \( \mu = 1 \), the result follows by Lemma 3.8. By Lemma 3.6 the only other possibility is \( \mu = 2 \). By Lemma 3.7, we have an \((8, 14, 7, 4, 3)\) BBIBD.

**EXAMPLE 4.2.** There exists an \((8, 14, 7, 4, 3)\) BBIBD which is a \((1, 2)\) BBIBD. This BBIBD is affine resolvable.

\[
\begin{array}{cccccccc}
1234 & 1256 & 1357 & 1458 & 2367 & 2468 & 3478 \\
5678 & 3478 & 2468 & 2367 & 1458 & 1357 & 1256 \\
\end{array}
\]

THEOREM 4.4. A BBIBD with \( k = 5 \) is a \((20t + 5, (4t + 1)(5t + 1), 5t + 1, 5, 1)\) BBIBD with \( m = 5t - 4, \mu = 1 \), or a \((15, 63, 21, 5, 6)\) BBIBD with \( m = 6, \mu = 2 \).

**Proof.** By Lemma 3.6, the only possibilities are \( \mu = 1 \) or \( \mu = 2 \). If \( \mu = 1 \), the parameters are given by Lemma 3.8.

Thus suppose \( \mu = 2 \). By Lemma 3.3, \( \lambda = \frac{r + 3}{4} \). By Lemma 3.5, \( m = \frac{3r - 15}{8} \). By Lemma 3.4, \( v = \frac{17r + 3}{r + 3} \). Thus we must have \( r \equiv 5 \mod 8 \). Also, \((r + 3)|17r + 3\), so \((r + 3)|48\). Thus \( r = 5, 13, 21, \text{ or } 45 \).

But, since \( 5|v \), the only possibility is \( r = 21 \). Thus the only possibility is a \((15, 63, 21, 5, 6)\) BBIBD, with \( m = 6, \mu = 2 \). Such a BBIBD is given in Example 4.3 below.

**EXAMPLE 4.3.** There exists a \((15, 63, 21, 5, 6)\) BBIBD which is a \((6, 2)\) BBIBD. The boss block is a \( b, c, d, e \).

\[
\begin{array}{cccccc}
a & b & c & d & e \\
0 & 1 & 1 & 2 & 1 & 3 & 4 & 1 \\
0 & 1 & 2 & 2 & 3 & 2 & 4 & 2 \\
\end{array}
\begin{array}{cccc}
2 & 1 & 3 & 1 \\
0 & 1 & 4 & 1 \\
\end{array}
\begin{array}{cccc}
0 & 2 & 3 & 2 \\
1 & 2 & 4 & 2 \\
\end{array}
\mod 5
\]

\[
\begin{array}{cccc}
c & d & e \\
0 & 1 & 2 & 3 & 2
\end{array}
\text{(mod 5)}
\begin{array}{cccc}
c & e & a \\
0 & 1 & 2 & 4 & 1
\end{array}
\text{(mod 5)}
\begin{array}{cccc}
a & c & b \\
0 & 1 & 2 & 2 & 3 & 2
\end{array}
\text{(mod 5)}
\begin{array}{cccc}
a & d & b \\
0 & 1 & 3 & 2 & 3 & 2
\end{array}
\text{(mod 5)}
\begin{array}{cccc}
a & b & d \\
0 & 1 & 1 & 2 & 4 & 2
\end{array}
\text{(mod 5)}
\]

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Similar methods can be used to prove the following.

THEOREM 4.5. The only possible BBIBDs with $k = 6$ or $k = 7$, and $\mu \geq 2$ are the following.

1. A $(1,3)$ BBIBD with parameters $(12, 22, 11, 6, 5)$
2. A $(16,2)$ BBIBD with parameters $(24, 184, 46, 6, 10)$.

An example of (1) is given in Section 6. The authors know of no example of (2).

We give the following theorem.

THEOREM 4.6. For any positive integer $k$ there exists only finitely many integers $v$ such that a BIBD $(v,b,r,k,\lambda)$ is an $(m,\mu)$ BBIBD, with $\mu > 1$.

Proof. By Lemma 3.6, $\mu \leq \frac{k}{2}$. By Lemma 3.4, $v = \frac{[(k-1)^2 + (\mu-1)]r}{(\mu-1)r + (k-\mu)}$.

Thus $v \leq \frac{(k-1)^2 + (\mu-1)}{(\mu-1)r}$

$\leq \frac{(k-1)^2}{(\mu-1)} + \frac{(k-\mu)}{(\mu-1)r} + 1$

Therefore with $k$ and $\mu$ fixed, $v$ is bounded, since $\mu \neq 1$. Thus there are only a finite number of possible values for $v$.

5. BBIBDs with $m \leq 3$.

It has been shown that a BBIBD with $m = 1$ has parameters $(n(n\lambda - \lambda + 1), n(n\lambda + 1), n\lambda + 1, n\lambda - \lambda + 1, \lambda)$ (see, for example [6]).

In this section we consider BBIBDs with $m = 2$ or $3$.

THEOREM 5.1. There exists no BBIBD with $m = 2$.

Proof. Suppose there exists a BBIBD with $m = 2$. By Lemma 3.1, $(r-2)|r(k-1)$, since $b$ is an integer. Thus if $r$ is odd $(r-2)|k-1$, and if $r$ is even, $\frac{r-2}{2}|(k-1)$.

If $r$ is odd, $r \leq k+1$. By Lemma 3.3, $\lambda \geq \mu$ since in any BIBD $r \geq k$. Then Lemma 3.2 implies $r \geq 2+k$. This is a contradiction.

If $r$ is even, then $\frac{r-2}{2}|(k-1)$, $r \geq 2+k$ implies $r-2 = 2(k-1)$, or $r = 2k$.

Then, by Lemma 3.4, $v = \frac{[(k-1)^2 + (\mu-1)]2k + (k-\mu)}{(\mu-1)2k + (k-\mu)}$.

Since $k|v$, thus $k|\mu$. However, $\mu < k$, so we have a contradiction.
Similar, but more complicated calculations can be used to obtain parameters of BBIBDs with $m = 3$. The following is obtained.

**Theorem 5.2.** The only BBIBD with $m = 3$ is a (6, 15, 5, 2, 1) BIBD.

**Remark.** The above design is a (3,1) BBIBD by Theorem 4.1.

**Proof.** From Lemma 3.1, we have $(r-3)|2r(k-1)$.

First, suppose $r \equiv 2$ or $4 \pmod{6}$. Then $(r-3)|(k-1)$, so $r \leq k-2$. But Lemma 3.2 implies $r \geq k+3$, a contradiction.

Next, suppose $r \equiv 1$ or $5 \pmod{6}$. Then $(r-3)|(2k-2)$, and as above $r \geq k+3$. Thus $r-3 = 2k-2$ so $r = 2k+1$. Substituting into Lemma 3.1 and using $bk = rv$, we obtain $v = \frac{k(4k+1)}{k+1}$. Thus $(k+1)|(4k+1)$, so $k = 2$.

This gives rise to the (6, 15, 5, 2, 1) BIBD.

If $r \equiv 0 \pmod{6}$, then $(\frac{r^2}{3} - 1)|(k-1)$. Since $r \geq k+3$, we have $r = 3k$ or $r = (3k+3)/2$. Proceeding as above, we obtain in the first case that $v = (5k-1)/2k$, so $k = 1$. In the second case, $v = k(7k+5)/(k+5)$, so $s = 1$, 5 or 13. These do not give rise to possible parameters for BIBDs.

Finally, we consider $r \equiv 3 \pmod{6}$. This case is handled similarly, and no BIBDs result.


As indicated in Section 2, an affine resolvable BIBD is a $(1, \frac{k^2}{v})$ BBIBD. However, the converse is not true, as the following example indicates.

**Example 6.1.** There exists a (12, 22, 11, 6, 5) BIBD which is a (1,3) BBIBD but not affine resolvable. The block $1 2 3 4 5 6$ can be taken as a boss block, and intersects all blocks except 7 8 9 10 11 12 in exactly three varieties. Yet this BIBD is not resolvable, since, for example, 1 2 3 7 8 9 occurs in no parallel class.

<table>
<thead>
<tr>
<th>1 2 3 4 5 6</th>
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<tbody>
<tr>
<td>2 3 5 7 10 12</td>
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<tr>
<td>2 3 5 8 9 11</td>
</tr>
<tr>
<td>1 2 3 7 8 9</td>
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<td>2 4 5 8 10 11</td>
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<td>3 5 6 7 8 12</td>
</tr>
<tr>
<td>1 2 6 7 8 12</td>
</tr>
<tr>
<td>1 2 6 9 10 11</td>
</tr>
</tbody>
</table>
It is clear that a \((1,\mu)\)BIBD is an affine resolvable BIBD if and only if every block is a boss block. However, as the above example indicates, not every block of a BIBD need be a boss block. In fact, it may be checked that 1 2 3 4 5 6 and 7 8 9 10 11 12 are the only boss blocks in the above BIBD.

In some cases, a BIBD has a unique boss block. Sufficient conditions for this to happen are given in the following theorem.

**Theorem 6.1.** A BIBD with \(m > 1\) and \(\mu\) not dividing \(k\) has a unique boss block.

**Proof.** Let \(B\) be a boss block. Any other block in the design intersects \(B\) in 0 or \(\mu\) varieties. Since \(\mu\) does not divide \(k\), any parallel class must contain \(B\). Since \(m > 1\) no block other than \(B\) can be a boss block.

In Example 4.2, \(m = 6, \mu = 2,\) and \(k = 5\). Thus Theorem 6.1 applies, and the block \(a b c d e\) is the only boss block in the design.

We close by describing an infinite class of boss block designs. This construction is due to Yul Inn [3].

**Theorem 6.2.** Let \(0 \leq t \leq n\) be integers. Then there exists a BIBD, with \(v = 2^{t+n} + 2^n - 2^n\), \(k = 2^t\) and \(\lambda = 1\), which is a \((2^t - 2^k + 1, 1)\) BIBD.

**Proof.** Denniston [2] has shown that if \(0 \leq t \leq n\) are integers, then there is a set of \(v = 2^{t+n} + 2^t - 2^n\) points in the projective plane of order \(2^n\) such that any line meets this set in either \(k = 2^t\) or 0 points. Such a set of points is called a \(\{2^t; 2^n\}\)-arc. Delete the points not in the arc, thus obtaining the desired BIBD.

Choose any block \(B\) as the boss block. Each of the points of the projective plane which were expunged from the line containing \(B\) induce a parallel class containing \(B\). There are \(m = 2^{n+1} - 2^k\) such points, and no two of the parallel classes formed contain a common block other than \(B\). The remaining blocks all meet \(B\) in \(\mu = 1\) point. Thus we have a boss block design.

7. Conclusion.

Thus we have generalized Bose's inequality to obtain

\[ b \geq v + r - 1 + \frac{r(m-1)(k-1)}{r-m} \]

for a BIBD containing a block \(B\) which occurs in \(m\) parallel classes no
two of which contain a common block other than \( B \). Designs which satisfy the above inequality as an equality are termed boss block designs and are studied for small values of \( k \) and \( m \).

REFERENCES


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