Linear Viscoelasticity and the Relaxation Spectrum

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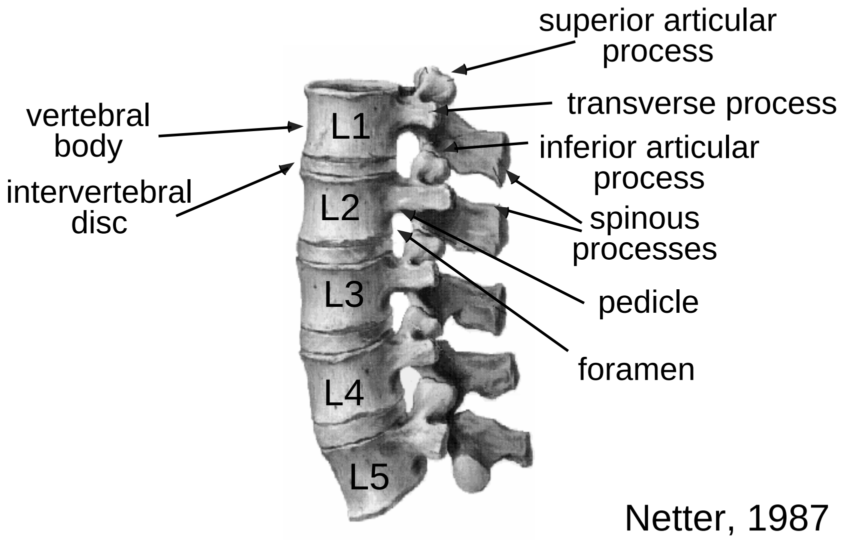
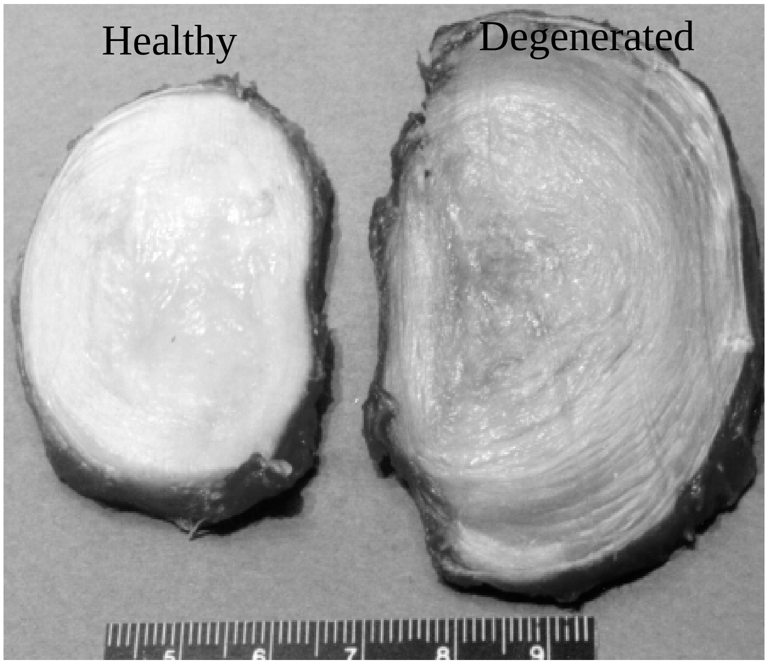
Lumbar Spine



Netter, 1987



Viscoelasticity

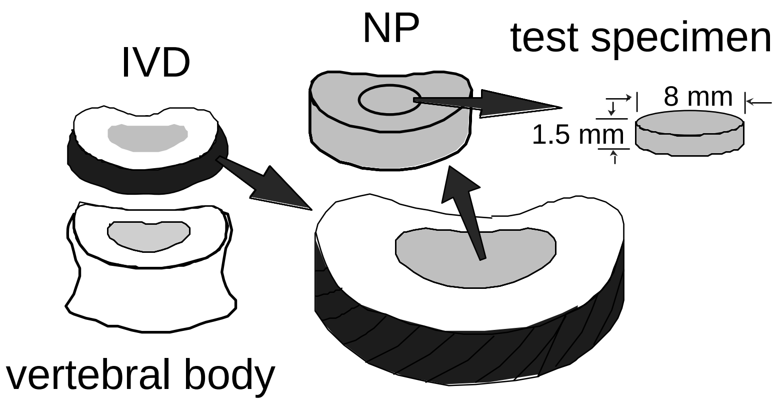
- Our bodies are mostly water
  - – it just makes sense that water affects the mechanical behavior of biological soft-tissues
- Viscoelasticity
  - – Flow independent viscoelasticity
    - polymeric matrix molecules reorient … at their own pace
    - No volume changes
  - – Flow dependent (Biphasic) viscoelasticity:
    - Water is forced out of the tissue like a sponge
    - Very small pores so this takes time
    - Requires volume changes

The Viscoelastic Shear Behavior of the Human Lumbar Nucleus Pulposus

Objectives

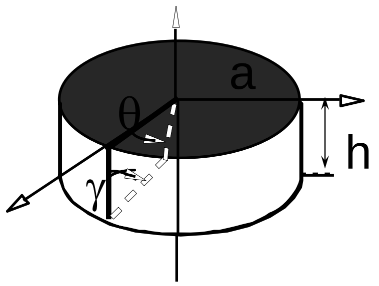
- Study the intrinsic viscoelastic behavior of the NP in shear
- Determine a constitutive relationship capable of describing this viscoelastic behavior

Specimen Preparation



Mechanical Testing

- Torsional shear strain (γ)

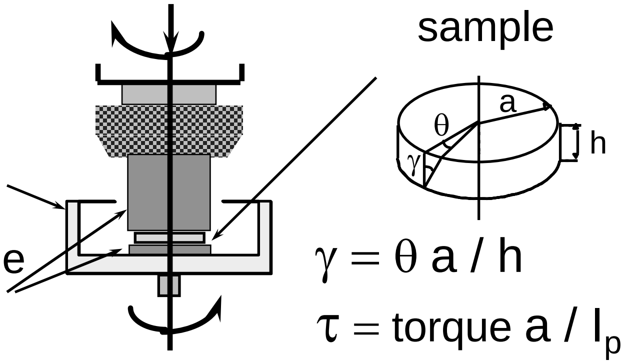


$$\gamma = \theta\, a / h$$

- No volumetric changes -->
  Negligible flow-dependent effects

Shear Testing Apparatus

normal force / torque



humidity chamber

impermeable platens

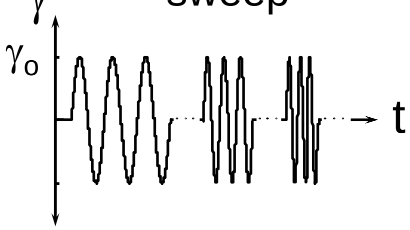
$\gamma = \theta a / h$

$\tau = \text{torque } a / I_p$

angular displacement

Shear Testing Protocol

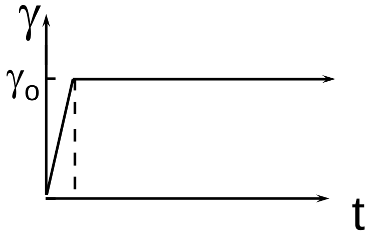
dynamic frequency sweep



$\gamma_o = 0.01$ rad

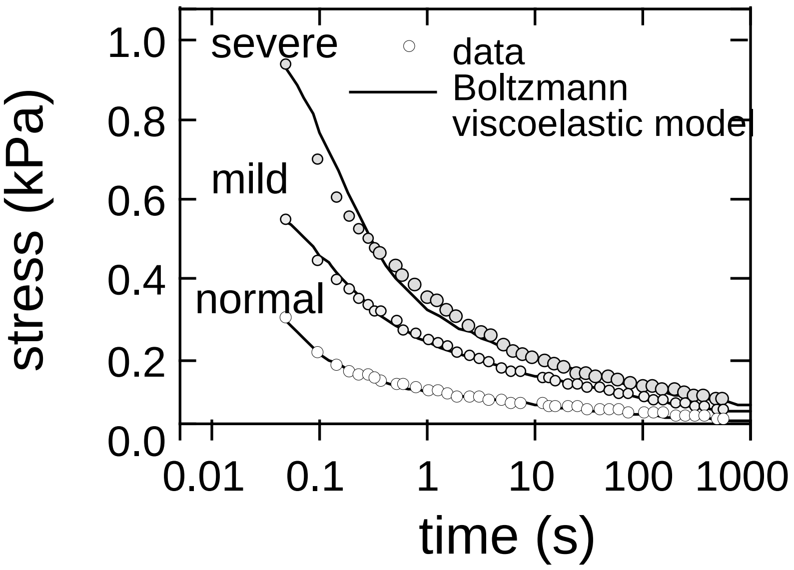
$1 < \omega < 100$ rad/s

stress relaxation experiments



3 tests @ $\gamma_o = 0.05$, 0.10, and 0.15 rad

Stress-Relaxation Experiment



Viscoelastic modeling

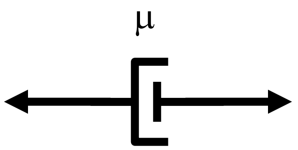
Elastic Solid & Viscous Fluid

- Elastic solid

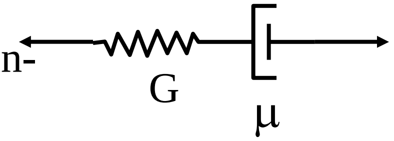
  $\tau = G\gamma$

- Viscous fluid

  $\tau = \mu\dot{\gamma}$



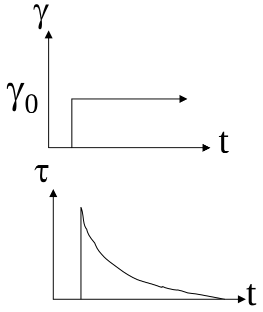
Viscoelastic Models
Maxwell Fluid: 2 constants



- Stress is a function of strain-rate

  $\tau + \lambda\dot{\tau} = \mu\dot{\gamma}$

  where $\lambda = \mu/G$ and is the time constant

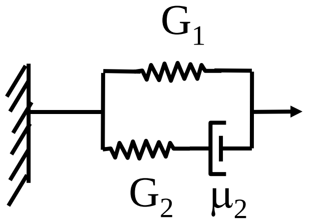


- Stress -relaxation function

$$\tau(t)/\gamma_0 = G(t) = (\mu/\lambda)\exp(-t/\lambda)$$

Relaxation function defined by viscosity μ and relaxation time λ

3 parameter solid

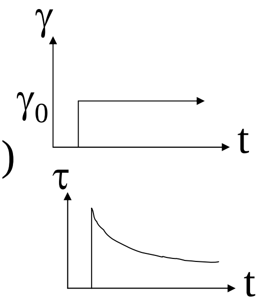


- Stress is a function of strain and strain-rate

$$\lambda_2\dot{\tau} + \tau = (\mu_2+\lambda_2 G_1)\dot{\gamma} + G_1\gamma$$

- Relaxation Function

$$\tau(t)/\gamma_0 = G(t) = G_1 + (\mu_2/\lambda_2)\exp(-t/\lambda_2)$$

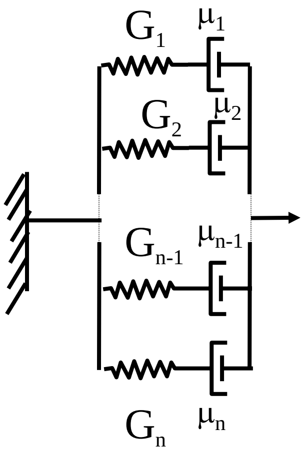


The relaxation function is defined by a single time constant, $\lambda=\mu/G$

Generalized Maxwell Model
2*n parameters

- Consists of ‘n’ Maxwell units in parallel
- Stress -relaxation function

$$G(t) = \Sigma(\mu_i/\lambda_i)\exp(-t/\lambda_i)$$

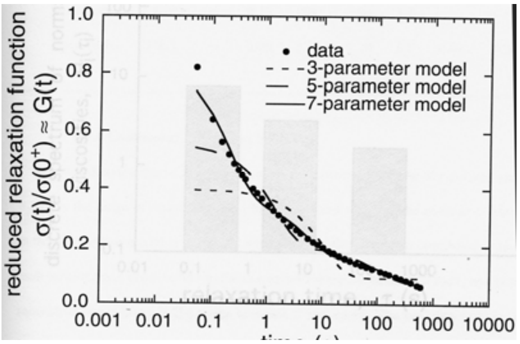


A relaxation function with DISCRETE relaxation times!

Experimental data for human nucleus pulposus

- Curve fit with generalized maxwell model to obtain magnitudes of the time constants

Good description of behavior!



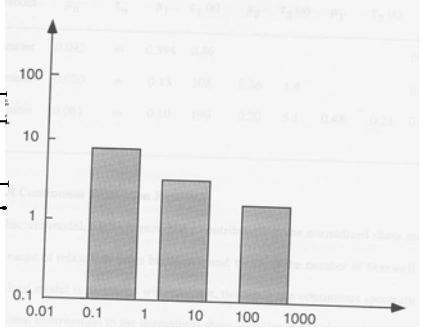
Discrete relaxation spectrum

- 7 material parameters
  - 3 viscosities
  - 3 time constants
  - 1 equilibrium shear modulus
- Each bar corresponds to a single relaxation time and viscosity
- Note that amplitude of viscosities decrease with relaxation time

Relaxation spectrum

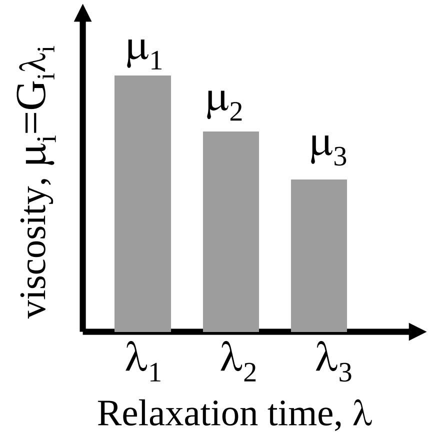
(only shows 6 parameters)

viscosity, $\mu_i = G_i \lambda_i$

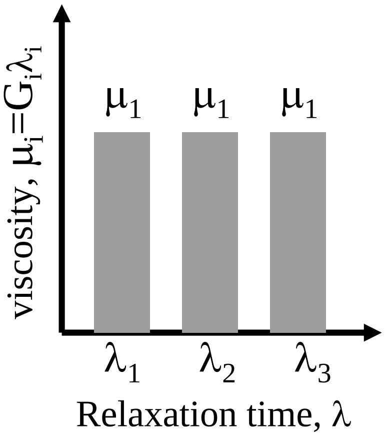


Relaxation time, λ_i

Discrete relaxation spectrum (2)



viscosities vary with relaxation time, 6 parameters needed

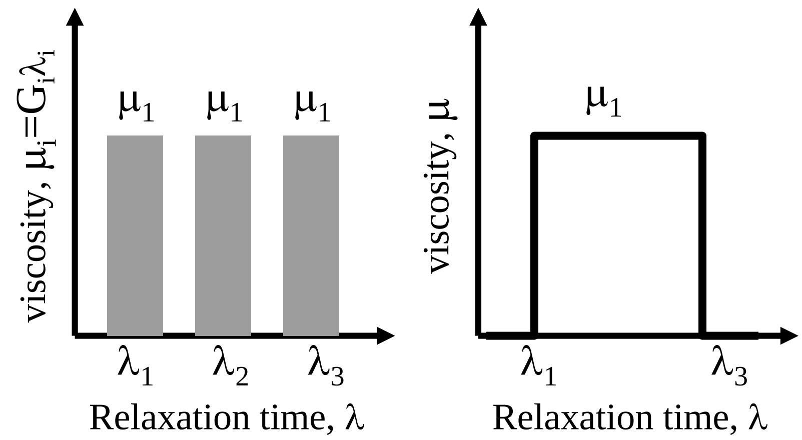


viscosities constant with relaxation time, 4 parameters needed

Discrete relaxation spectrum (3)

- If a material has viscosities that <u>are not</u> a function of relaxation time, it is called a strain rate insensitive material
- If a material has has viscosities that <u>are</u> a function of relaxation time, it is called a strain rate sensitive material

Discrete & Continuous Relaxation Spectra



Discrete spectrum,
4 parameters

Continuous spectrum,
3 parameters

Continuous Relaxation function

- An integral formulation for a continuous spectrum of relaxation times with constant amplitude

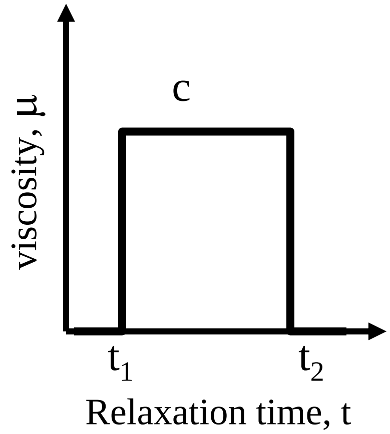
$$G(t) = 1 + \int_0^\infty S(t') \exp(-t/t')dt'$$

$$S(t) = \begin{cases} c/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

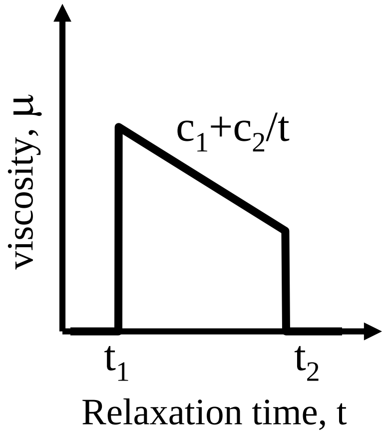
Fung, 1981

Only 3 constants: c, t1, t2
these are similar to $\mu 1$, λ_1 and λ_3

Continuous Relaxation Spectra



Continuous spectrum,
3 parameters



Continuous spectrum,
4 parameters

Continuous relaxation spectrum viscositites that decrease with relaxation time

Integral formulation with viscosities that decrease with relaxation time

$$G(t) = 1 + \int_0^\infty S(t') \exp(-t/t')\,dt'$$

Fung, 1981

Variable amplitude spectrum

$$S(t) = \begin{cases} (c_1 + c_2/t)/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

Iatridis et al., 1997

Reduced relaxation function

- Normalized by the value at t=0
- For a discrete relaxation spectrum, we have:

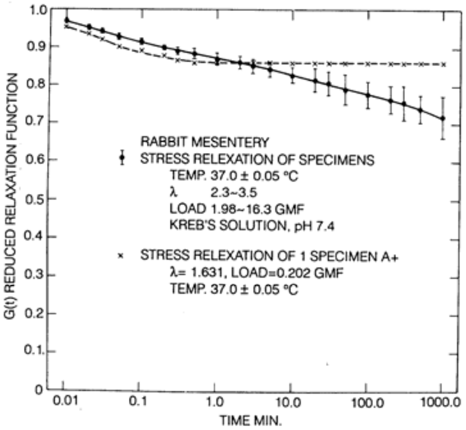
$$G(t) = \frac{\sum C_i e^{-t/\lambda_i}}{\sum C_i}$$

Reduced relaxation function

- 2 important points
  - if an experiment is cut off prematurely, one may mistakenly arrive at the limiting value G(∞), corresponding to $\lambda_i = \infty$
  - the relaxation times, λ_i should not be interpreted literally without realization that representation of empirical data by a sum of exponentials is a non-unique process

Defining appropriate equilibrium time

- Stress relaxation for an individual specimen ended at 1min, however, for the average it went beyond 1000 min
- Fung, Fig7.5:4



Non-uniqueness of exponentials

- These 3 exponentials all fit a certain set of experimental data equally well for x between 0 and 1. According to Lanczos (1956) and taken from Fung, p. 280.

$$f(x) = 2.202e^{-4.45x} + 0.305e^{-1.58x},$$
$$f(x) = 0.0951e^{-x} + 0.8607e^{-3x} + 1.5576e^{-5x},$$
$$f(x) = 0.041e^{-0.5x} + 0.79e^{-2.73x} + 1.68e^{-4.96x}.$$

Reduced relaxation function with variable amplitude spectrum

Reduced relaxation function

reduced means normalized by the value at t=0

$$G(t) = \frac{1 + \int_0^\infty S(t')\exp(-t/t')dt'}{1 + \int_0^\infty S(t')\,dt'}$$

Fung, 1981

Variable amplitude spectrum

$$S(t) = \begin{cases} (c_1 + c_2 / t)/t & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

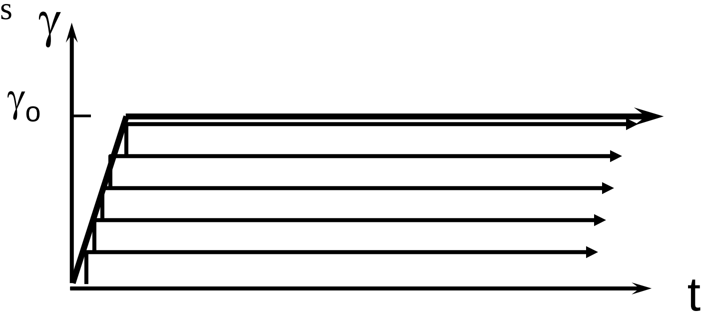
Iatridis et al., 1997

Integral formulation for linear viscoelastic models

- Now that we know how to describe a step stress-relaxation experiment, we are ready to figure out how to describe any type of loading experiment

Ramp stress-relaxation experiment

- A ramp stress relaxation experiment can be considered to be comprised of a series of infinitesimally small step stress relaxation experiments



- Any (displacement controlled) experiment can be considered to be comprised of a series of step stress relaxation summed together appropriately

Boltzmann Linear Viscoelasticity Model (for stress response)

- Boltzmann superposition: Integral formulation for loading modes other than step strain input
- That is the superposition of infinite number of step stress relaxation experiments

$$\tau(t) = \int_{-\infty}^{t} G(t - t')\, G\, \frac{d\gamma}{dt'}\, dt'$$

$\tau(t)$ = shear stress

$\gamma(t)$ = shear strain

$G(t)$ = reduced relaxation function

G = instantaneous shear modulus

Boltzmann Linear viscoelasticity

- If the motion starts at t=0 and $\tau = \gamma = 0$ for $t<0$, we have the following (more practical) relationship:

$$\tau(t) = G(0)\mathrm{G}\gamma(t) + \int_{0}^{t} \dot{G}(t')\mathrm{G}\gamma(t-t')dt'$$

but G(0)=1

- therefore, the shear stress at time t is equal to the instantaneous stress response decreased by an amount depending on the past history because dG/dt' is usually negative

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Boltzmann Linear Viscoelasticity Model (for strain response)

•Boltzmann superposition can also be used for creep responses…. That is the superposition of infinite number of step creep experiments

$$\gamma\,(t) = \int_{-\infty}^{t} J(t - t')\ \mathrm{J}\ \frac{d\tau}{dt'}dt'$$

τ (t) = shear stress
γ(t) = shear strain
J(t) = reduced creep function
J = instantaneous shear compliance

Model predictions

- Use the Boltzmann linear viscoelasticity model (for stress response) with variable amplitude and continuous relaxation spectrum to describe and predict material behaviors

Boltzmann Viscoelastic Law

$$\tau\,(t) = \int_{-\infty}^{t} G(t - t') \frac{d\gamma}{dt'} \mu, \quad dt'$$

τ (t) = shear stress

γ(t) = shear strain

G(t) = reduced relaxation function

μ = instantaneous shear modulus

Relaxation Function

Reduced relaxation function

$$G(t) = \frac{1 + \int_0^\infty S(t') \exp(-t/t')dt'}{1 + \int_0^\infty S(t')\, dt'}$$

Fung, 1981

Variable amplitude spectrum

$$S(t) = \begin{cases} c_1/ t + c_2 / t^2 & t_1 < t < t_2 \\ 0 & \text{otherwise} \end{cases}$$

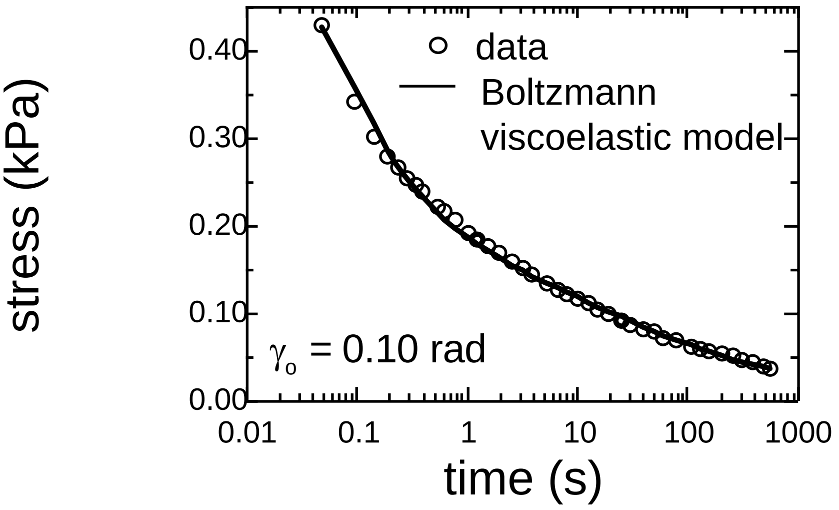
Materials and Method

- 19 human lumbar discs (L2-5)
  average age 57 yrs (16-88 yrs)
- cylindrical NP specimens
  1.78 mm thick x 8.0 mm dia
- samples were prepared while frozen and tested in a humidity chamber

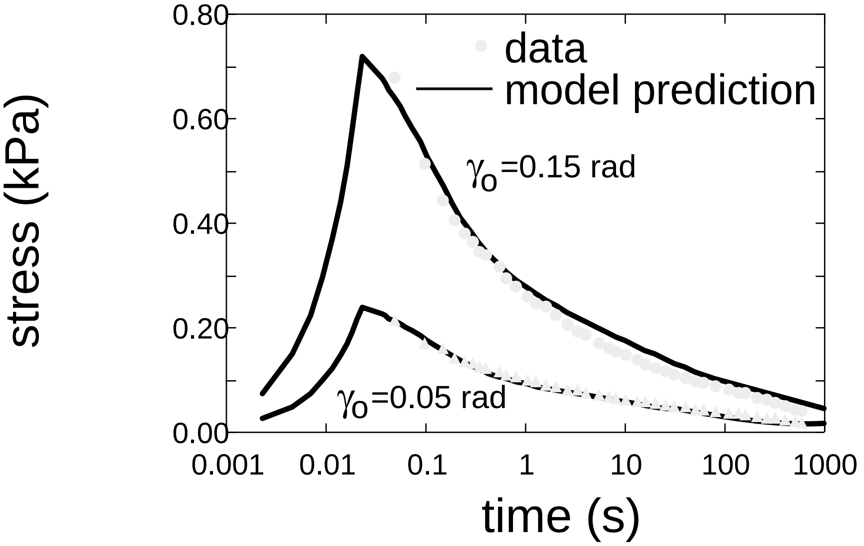
Degenerative Level of NP

normal	mild	severe
grade I (n=7)	grade II-III (n=5)	grade IV-V (n=7)
gel-like blue-white	consolidation of fibrous tissue	clefts fissures

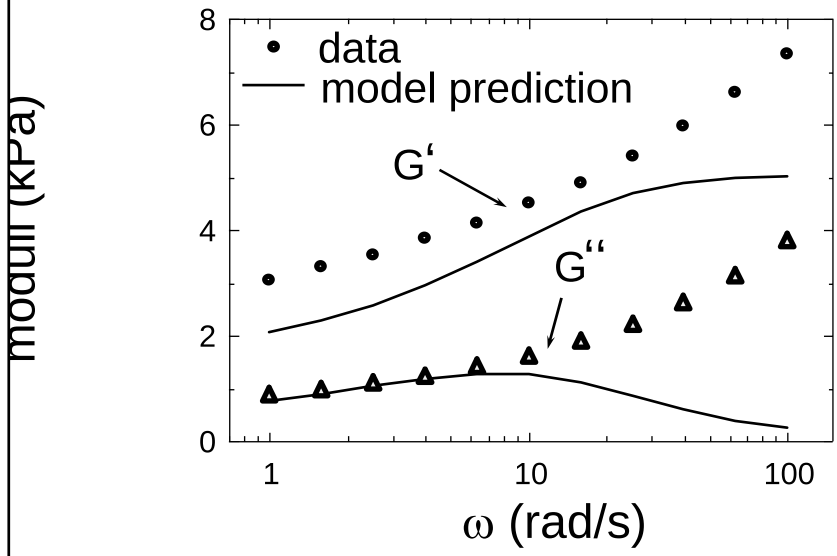
Stress Relaxation



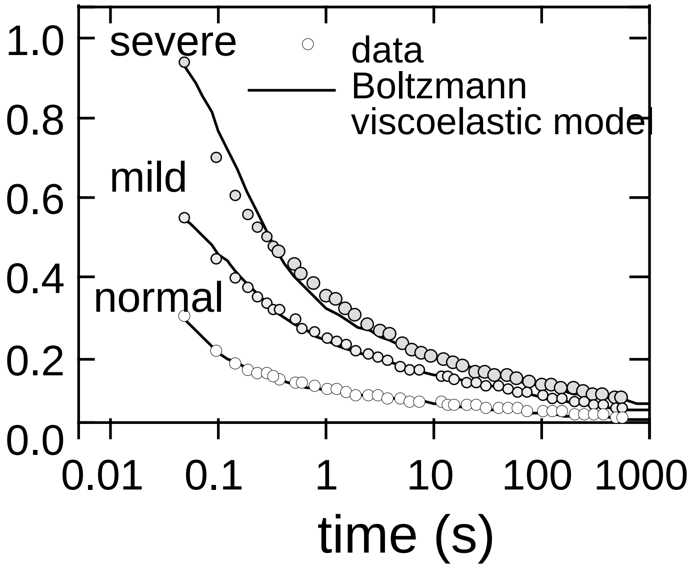
Stress Relaxation



Dynamic Frequency Sweep



Stress-Relaxation Experiment

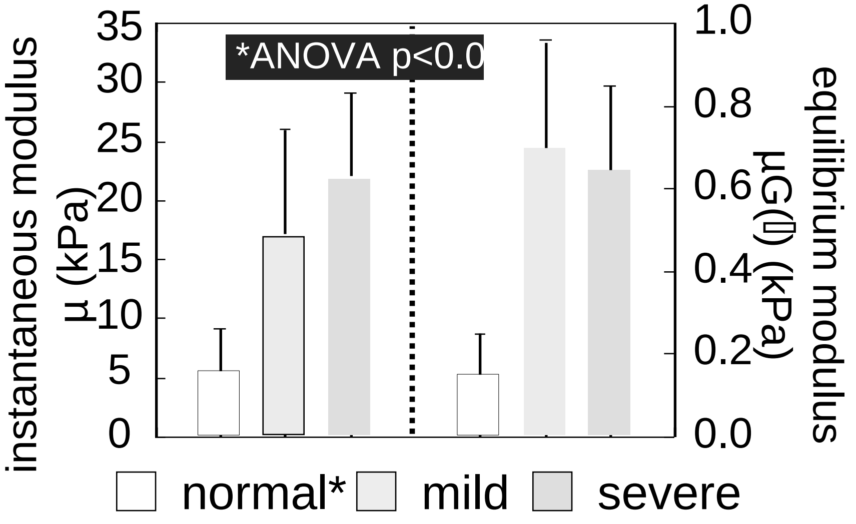


Model Parameters

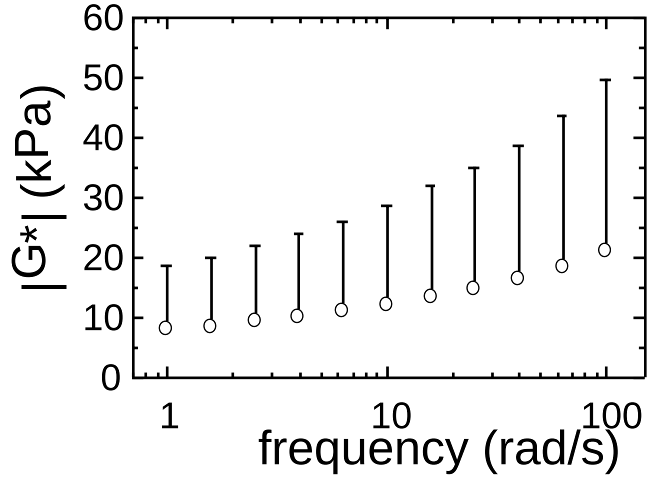
	normal	mild	severe
μ (kPa)*	5.8±3.7	17.1±10.1	22.5±7.8
c_1	1.3±0.6	1.0±0.4	0.9±0.3
c_2 (s)	1.0±0.6	1.2±0.5	1.2±0.7
τ_1 (s)	0.05±0.02	0.09±0.08	0.04±0.02
τ_2 (s)	600	600	600

* significant effect of degeneration

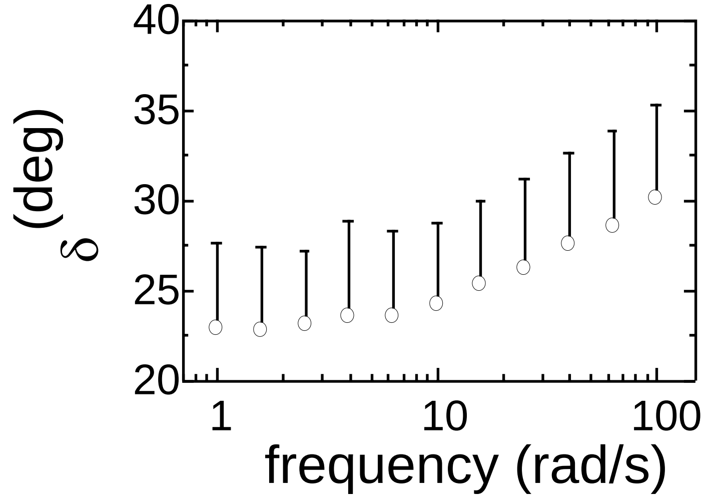
Shear Moduli



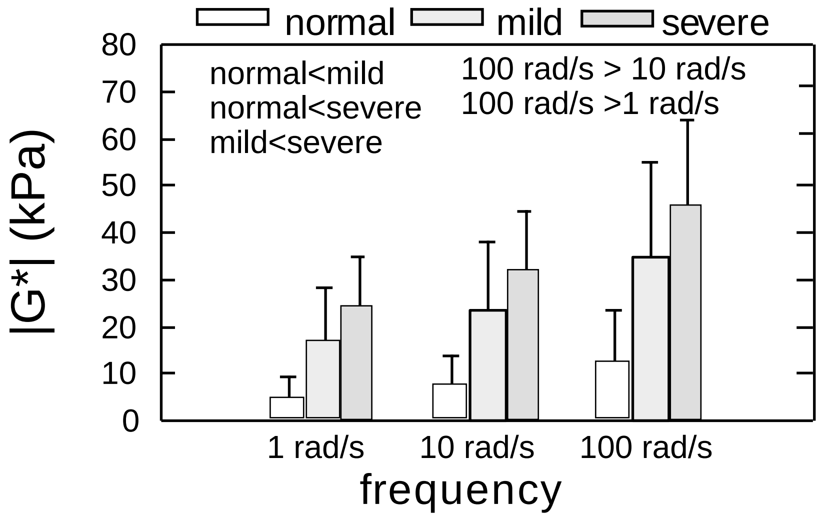
Dynamic Frequency Sweep



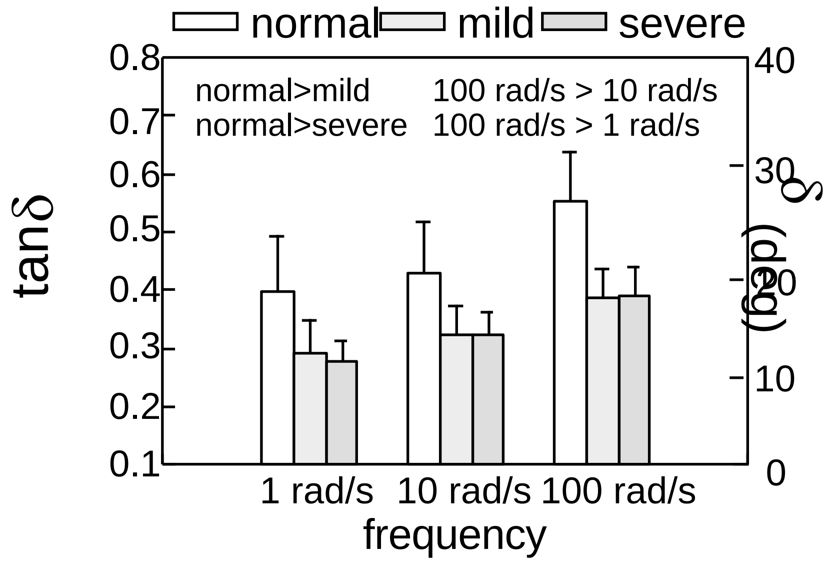
Dynamic Frequency Sweep



Dynamic Experiment



Dynamic Experiment



Dynamic Material Functions

	$\lvert G^* \rvert$ (kPa)	δ (deg)	
articular cartilage	500	15	solid
meniscus	100	22	
degenerate NP	60	17	
normal NP	11	24	
proteoglycan solution	0.01	65	fluid

parameters determined @ $\omega = 10$ rad / s

Closure

This lecture summarized 3 major concepts:

1) The concept of relaxation spectrum and the pros/cons of choosing discrete vs continuous forumlations. Note that the relaxation function is the solution for a step stress relaxation experiment.
2) The Boltzmann integral formulation for linear viscoelasticity was introduced and this was presented to provide a general way in which the relaxation function could applied to general displacement control experiments, e.g., ramp stress relaxation and sinusoudal tests
3) One specific and somewhat complex form for linear viscoelastic model (i.e., the variable amplitude and continuous relaxation spectrum applied using the Boltzmann formulation for general loading conditions) was applied to the shear behaviors of human nucleus pulposus tissues.

Future directions

- The next topic will be to next topic will be to expand on these concepts to analyze quasi-linear viscoelasticity and then to apply some of these concepts to experimental data.