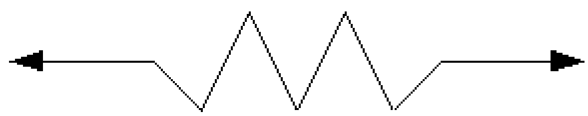
Biomechanics II: Generalized Viscoelastic Materials
1/23/08

Key Points:
1) Build a viscoelastic constitutive model from spring and damping elements.
2) Apply boundary conditions appropriate to a physical scenario.
3) Solve the differential equation.
4) Have some idea of what the result should look like.

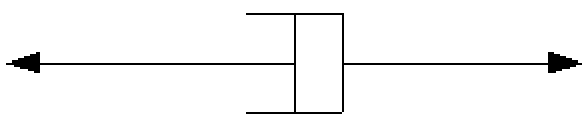
Conventions:
1) σ is stess, ε is strain. For descriptions of structural behavior, force and displacement are more appropriate.
2) Capitalized symbols denote Laplace Transforms.

Basic Elements:
Spring (like a resistor)



$$\sigma = R\varepsilon$$

Dashpot (like a capacitor)



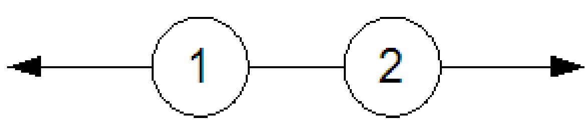
$$\sigma = \eta\dot{\varepsilon}$$

Of possibly more interest than the stiffness of these elements is the compliance, which is given by $\varepsilon = 1/R\sigma$ for the spring and $\varepsilon = 1/\eta\int_0^t \sigma(\tau)\,d\tau$ for the dashpot. Note that while the strain of the spring is affected only by the current state of stress, the strain of the dashpot is determined by its entire stress history.

Combining Elements

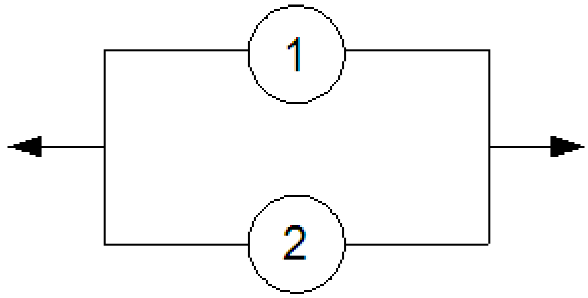
In series, the total strain of the system is equal to the sum of the strain in the individual components. Total stress is equal to the stress in each component:

$$\varepsilon = \varepsilon_1 + \varepsilon_2 \ , \ \sigma = \sigma_1 = \sigma_2$$



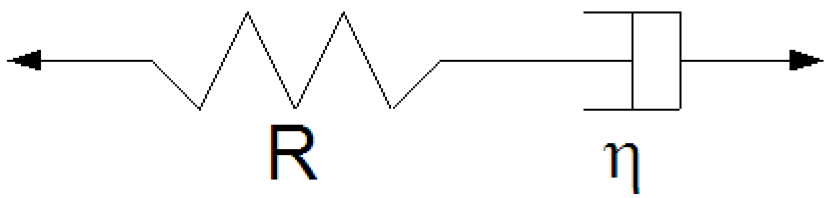
In parallel, all displacements are equal, and component stresses add:

$$\varepsilon = \varepsilon_1 = \varepsilon_2, \sigma = \sigma_1 + \sigma_2$$



From these basic rules, a governing differential equation may be constructed from combinations of spring and dashpot elements. These models range from very basic to very complex, but can all be classified as solid or fluid models. Fluid models are defined as arrangements in which deformation induced by an applied stress will not recover after the stress is removed.

Basic Fluid, Maxwell Model



$$\sigma_1 = R\varepsilon_1, \sigma_2 = \eta\dot{\varepsilon}_2$$
$$\varepsilon = \varepsilon_1 + \varepsilon_2$$
$$\dot{\varepsilon} = \dot{\varepsilon_1} + \dot{\varepsilon}_2$$
$$\sigma = \sigma_1 = \sigma_2$$

Combining:

$$\dot{\varepsilon} = \dot{\varepsilon_1} = \dot{\varepsilon}_2 = \frac{1}{R}\dot{\sigma} + \frac{1}{\eta}\sigma$$

More commonly:

$$\sigma + \frac{\eta}{R}\dot{\sigma} = \eta\dot{\varepsilon}$$

Laplace Transform of the governing equation:

$$\Sigma + \frac{\eta s}{R}\Sigma = \eta s E$$

Steady state creep with normalized time $\sigma = \sigma_0 u(0), \Sigma = \frac{\sigma_0}{s}, \sigma(0) = \varepsilon(0) = 0$

Substituting into transformed governing equation

$$\left(\frac{1}{s} + \frac{\eta}{R}\right)\sigma_0 = \eta s \Sigma$$

$$\Sigma = \sigma_0 \frac{R + \eta s}{\eta s^2 R}$$

Inverting the transform

$$\varepsilon=\sigma_0\left(\frac{1}{R}+\frac{t}{\eta}\right)$$

Result: Initial elastic step followed by constant rate flow.

Stress-Relaxation with normalized time $\varepsilon=\varepsilon_0 u(0), E=\frac{\varepsilon_0}{s}, \sigma(0)=0, \varepsilon(0)=0$

Substituting into the transformed governing equation

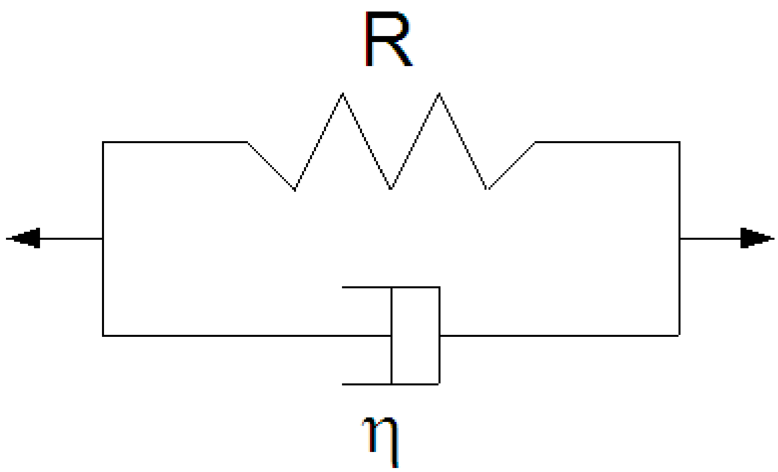
$$\Sigma=\varepsilon_0\frac{\eta R}{R+\eta s}=\varepsilon_0 R\left(\frac{1}{R/\eta+s}\right)$$

Inverting the transform:

$$\sigma=\varepsilon_0 Re^{\frac{-Rt}{\eta}}$$

Result: Initial elastic step followed by exponential decay to zero.

Basic Solid, The Kelvin Model



$$\sigma_1=R\varepsilon_1, \sigma_2=\eta\dot{\varepsilon}_2$$
$$\varepsilon=\varepsilon_1=\varepsilon_2$$
$$\sigma=\sigma_1+\sigma_2$$

Applying Conditions

$$\sigma=R\varepsilon+\eta\dot{\varepsilon}$$

The Laplace Transform of the governing equation

$$\Sigma=RE+\eta sE$$

Static Creep with normalized time $\sigma=\sigma_0 u(0), \Sigma=\frac{\sigma_0}{s}, \sigma(0)=\varepsilon(0)=0$

Substituting into the transformed governing equation, and re-arranging:

$$\frac{\sigma_0}{s}=\Sigma(R+\eta s)$$

$$\Sigma=\frac{\sigma_0}{s(R+\eta s)}=\frac{\sigma_0}{\eta}\frac{1}{s(s+R/\eta)}$$

$$\Sigma=\frac{\sigma_0}{R}\left[\frac{R/\eta}{s(s+R/\eta)}\right]$$

Inverting the transform:

$$\varepsilon=\frac{\sigma_0}{R}\left(1-e\frac{-Rt}{\eta}\right)$$

Result: Displacement starts at zero and approaches elastic steady state as a decaying exponential.

Stress-Relaxation with normalized time $\varepsilon=\varepsilon_0 u(0), E=\frac{\varepsilon_0}{s}, \sigma(0)=0, \varepsilon(0)=0$

Substituting into the transformed governing equation and re-arranging:

$$\Sigma=\varepsilon_0\frac{R+\eta s}{s}$$

Here's where things get tricky. If we invert this transform, we get:

$$\sigma=\varepsilon_0(\eta+R\delta(t))$$

The Dirac delta function shows up because we imposed our displacement boundary using a Heaviside step function at time zero. Recall from the governing equation that the stress in the damping element is proportional to the rate of strain, which in this case is infinite upon application. This is an important distinction when we model stress relaxation experiments performed out in the real world, where displacements are applied at a finite rate. In these cases, we need to solve for the stress using Boltzman superposition, and account for both ramp and relaxation periods.