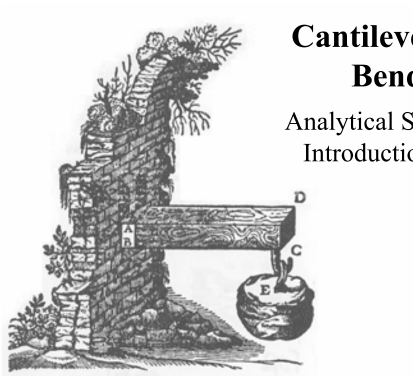


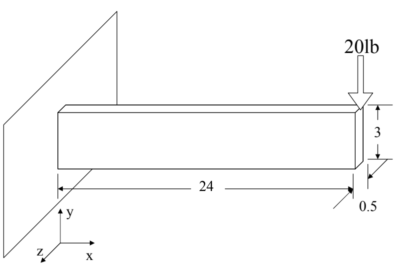
Cantilever Beam Bending

Analytical Solution and Introduction to FEA

Lab Exercises

- Experimental measurement of force vs. deflection curves of brass and aluminum U channel subjected to three point bending
- Analytical predictions
- Finite Element Analysis with comparison to analytical results
  - -1D & 2D
  - - Mesh size and element type

Cantilever Beam with End Load



Cantilever Beam with End Load

Euler Beam subjected to a bending moment:

$$\frac{M(x)}{EI} = \frac{\frac{d^2\delta}{dx^2}}{\left[1+\left(\frac{d\delta}{dx}\right)^2\right]^{3/2}}$$

$$\frac{d\delta}{dx} \to 0 \qquad \frac{M(x)}{EI} = \frac{d^2\delta}{dx^2} \qquad M(x) = EI\frac{d^2\delta}{dx^2}$$

$$\sigma_{axial} = \frac{My}{I}$$

For a rectangular cross section: $I = \frac{bh^3}{12}$

Cantilever Beam with End Load

For a point load at $x=l$

$$M(x) = Fl - Fx$$

$$\frac{1}{EI}(Fl - Fx) = \frac{d^2\delta}{dx^2}$$

$$\frac{1}{EI}\left(Flx - \tfrac{1}{2}Fx^2 + C_0\right) = \frac{d\delta}{dx}$$

$$\frac{1}{EI}\left(\tfrac{1}{2}Flx^2 - \tfrac{1}{6}Fx^3 + C_0x + C_1\right) = \delta$$

Applying Boundary Conditions:

$$\delta\big|_{x=0} = 0 \qquad \frac{d\delta}{dx}\bigg|_{x=0} = 0$$

$$C_0 = C_1 = 0$$

$$\frac{1}{EI}\left(\tfrac{1}{2}Flx^2 - \tfrac{1}{6}Fx^3\right) = \delta$$

Cantilever Beam with End Load

$E = 29\times10^6$

$I = 1.125$

$F = -20$

$l = 24$

$$\sigma_{axial} = \frac{(Fl - Fx)y}{I}$$

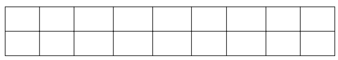
$$\frac{1}{(29\times10^6)(1.125)}\left(\tfrac{1}{2}(-20)24x^2 - \tfrac{1}{6}(-20)x^3\right) = \delta$$

$$\frac{1}{(3.2625\times10^6)}\left(\tfrac{10}{3}x^3 - 240x^2\right) = \delta$$

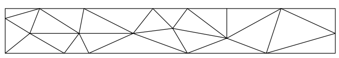
$$\boxed{\sigma_{axial}\big|_{x=0} = 576\text{psi}} \qquad \boxed{\delta\big|_{x=l} = -2.825\times10^{-3}}$$

Finite Element Mesh

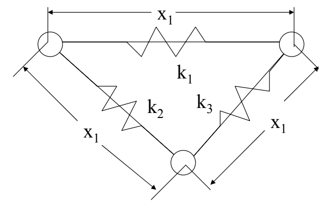
Structured - Quad



Unstructured - Tri



Finite Element Mesh



$$F_i = k_i \Delta x_i$$

Matrix Equation

$$\{F\} = [k]\{d\} \quad \Longrightarrow \quad \begin{Bmatrix} F_s \\ F_u \end{Bmatrix} = \left[\begin{array}{c|c} k_{ss} & k_{su} \\ \hline k_{us} & k_{uu} \end{array}\right] \begin{Bmatrix} d_u \\ d_s \end{Bmatrix}$$

Stiffness Matrix

Where s = known and u = unknown

$$\{F_u\} = [k_{us}]\{d_u\} + [k_{uu}]\{d_s\}$$

$$\{d_u\} = [k_{ss}]^{-1}\left(\{F_s\} - [k_{su}]\{d_s\}\right)$$

! $[k_{ss}]^{-1}$ Only exists for statically determinant systems! **!**

Forming the Stiffness Matrix

Galerkin Integration: Introduced in 1915 by Boris Grigorievich Galerkin

Given governing equation: $L\left(u(x)\right) = 0$ Where *u(x)* is unknown

u(x) may be approximated by a known function, *w(x)*, with variable parameters

Replacing *u(x)* with *w(x)*, the error of the substitution may be defined by:

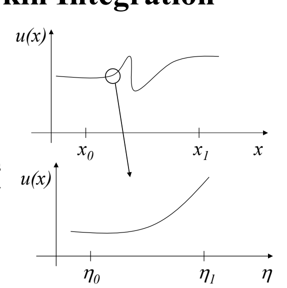
$$\int_{x_0}^{x_1} w(x) L\left(w(x)\right) dx$$

With the appropriate parameters in *w(x)*:

$$\int_{x_0}^{x_1} w(x) L\left(w(x)\right) dx = 0$$

Discrete Galerkin Integration

This integral will only yield the exact solution if there exists a function *w(x)* that is readily guessed and can equal *u(x)* over the entire interval.



However, if the coordinate system is transformed such that $x \circ \eta$, where η_1-η_0 is a small, discrete piece of *x*,

$$\int_{\eta_0}^{\eta_1} w(\eta) L\left(w(\eta)\right) d\eta = 0$$

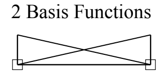
As the length of interval η_1-η_0 approaches zero, the exact solution is obtained over that interval regardless of the form of $w(\eta)$. By performing this calculation over a finite series of intervals spanning x_1-x_0, an approximation of the exact solution *u(x)* is obtained.

Shape Functions

1-D

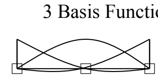
Linear: 2 Nodes

2 Basis Functions



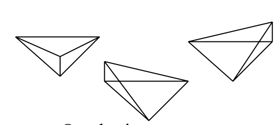
Quadratic: 3 Nodes

3 Basis Functions

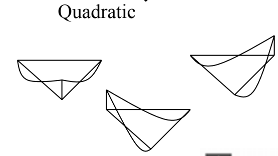


2-D

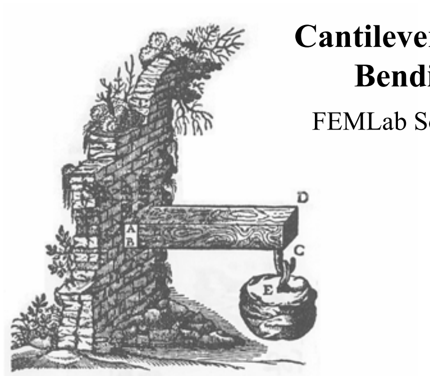
Linear



Quadratic



Cantilever Beam Bending



FEMLab Solutions

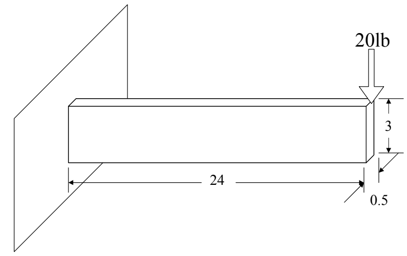
Cantilever Beam with End Load

$$\delta = \frac{1}{EI}\left(\tfrac{1}{2}Flx^2 - \tfrac{1}{6}Fx^3\right) \qquad \delta_{max} = \delta(l) = \frac{Fl^3}{3EI}$$

$E = 29\times10^6$

$I = 1.125$

$l = 24$



$\delta_{max} = -2.825\times10^{-3}$

1-D FEMLab Model

1-D Finite Element Model in FEMLab

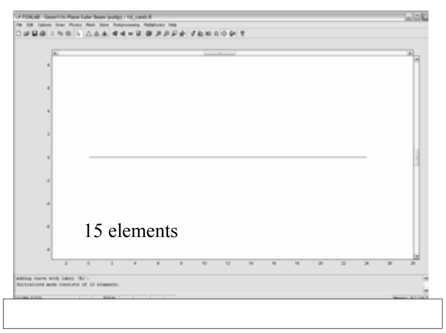
Assumptions:

- The beam has constant cross section
- Stress distribution is linear through height of the beam
- 1-D elements lie along the neutral axis of the beam

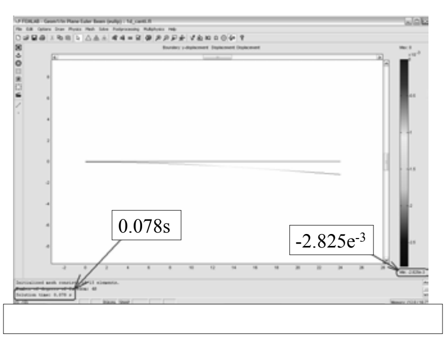
Boundary Conditions:

- $R_x = R_y = R_z = 0$ at $x = 0$
- $F_y = -20$ lb at $x=l$

1-D FEMLab Model



1-D FEMLab Model



2-D FEMLab Model

2-D Finite Element Model in FEMLab

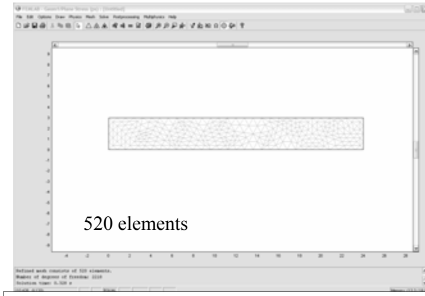
Assumptions:

- The beam has constant thickness
- Stress distribution is constant through the thickness of the plate
- There are no out of plane deformations
- 2-D elements lie along the center plane of the plate

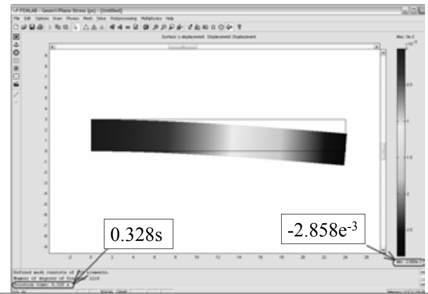
Boundary Conditions:

- $R_x = R_y = R_z = 0$ on $x = 0$, $-h/2 < y < h/2$
- $F_y = -20$ lb at $x = l$, $y = h/2$

2-D FEMLab Model



2-D FEMLab Model



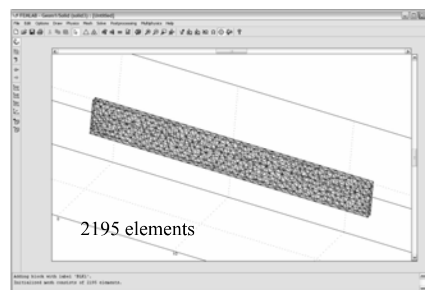
3-D FEMLab Model

3-D Finite Element Model in FEMLab

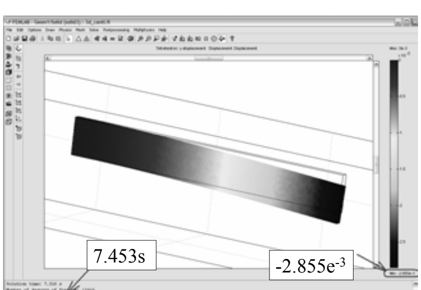
Assumptions:

- Material is linearly elastic and isotropic
- Boundary Conditions:
  - $R_x = R_y = R_z = 0$ on $x=0$, $-h/2 < y < h/2$, $-t/2 < z < t/2$
  - $F_y = -20$ lb/in at $x = l$, $y = h/2$, $-t/2 < z < t/2$

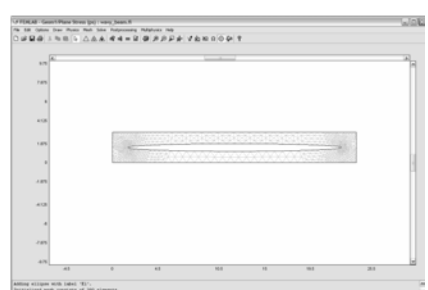
3-D FEMLab Model



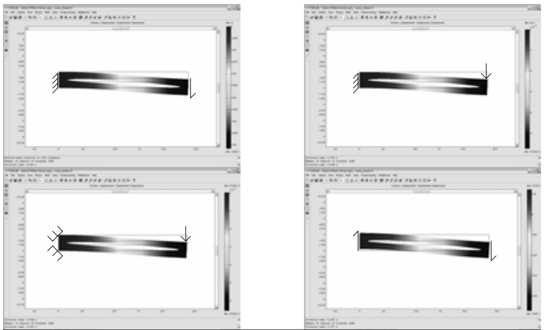
3-D FEMLab Model



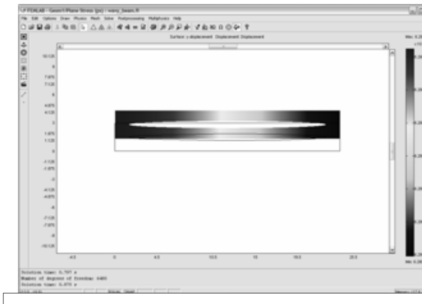
Complex Geometries



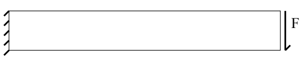
Selection of Boundary Conditions



Selection of Boundary Conditions

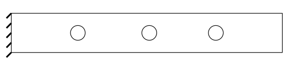


Design Applications



Von Mises Stress

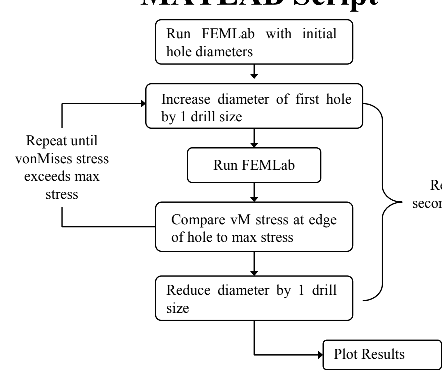
$\sigma_{\max} \approx 1800\, psi$



How big to drill the holes?

MATLAB M-File

MATLAB Script



Design Applications