The Sweep Map

Drew Armstrong — University of Miami
Nick Loehr — Virginia Tech & US Naval Academy
Greg Warrington — University of Vermont

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An injective sorting map

Area is the only statistic
A typical sort

\textbf{NENNEE}

Assign a weight to each letter:

\[ \text{wt}(N) = +1, \quad \text{wt}(E) = -1. \]

Sort in decreasing order of weight.
A typical sort

*NENNEE* $\rightarrow$ *NNNEEEEE*

Assign a weight to each letter:

$$\text{wt}(N) = +1, \quad \text{wt}(E) = -1.$$ 

Sort in decreasing order of weight.
Define a Dyck path of order $n$ to be a NE-lattice path from $(0, 0)$ to $(n, n)$ that stays weakly above $y = x$.

Let $w = w_1 \cdots w_n \in \mathcal{D}_n$.

Define levels

$$
\ell_i = \ell_i(w) = \begin{cases} 
0, & i = 0, \\
\ell_{i-1} + \text{wt}(w_i), & i > 0.
\end{cases}
$$
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, −1, −2, \ldots
Break ties from Right to Left.
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots

Break ties from Right to Left.
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots

Break ties from Right to Left.

\[ y = x + 4 \]

\[
\begin{array}{c}
2 \\
1 \\
1 \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
0 \\
1 \\
0 \\
\end{array}
\]

\[
\begin{array}{c}
1 \\
0 \\
1 \\
2 \\
\end{array}
\]
Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots
Break ties from \textbf{Right to Left}.

\[ y = x + 3 \]
Sorting by level

Sort steps by levels: \( \ldots, 2, 1, 0, -1, -2, \ldots \)

Break ties from Right to Left.

\[ y = x + 2 \]
Sorting by level

Sort steps by levels: . . . , 2, 1, 0, −1, −2, . . .

Break ties from Right to Left.
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots

Break ties from Right to Left.

\[ y = x + 1 \]
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, −1, −2, \ldots

Break ties from Right to Left.

\[ y = x + 1 \]
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots

Break ties from Right to Left.

\[ y = x + 0 \]
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, -1, -2, \ldots

Break ties from Right to Left.
Sorting by level

Sort steps by levels: \ldots, 2, 1, 0, \ldots, 1, -1, -2, \ldots
Break ties from Right to Left.
Order-3 Dyck paths
Order-3 Dyck paths

Theorem[L ’03]: The map defined is a bijection on Dyck paths of order $n$. 
Oops
All is not lost

A proper sorting order for levels:

\[-1, -2, -3, \ldots, \ldots, 2, 1, 0.\]
All $2 \times 2$ paths
Rational variations
Rational variations
The sweep map $\text{sw}_{\text{wt}}$

An alphabet $A = \{x_1, \ldots, x_k\}$,
A weight function $\text{wt} : A \rightarrow \mathbb{Z}$,
A word $w = w_1w_2 \cdots w_n \in A^*$
and levels

$$\ell_i = \ell_i(w) = \begin{cases} 
0, & i = 0, \\
\ell_{i-1} + \text{wt}(w_i), & i > 0.
\end{cases}$$
Define $\mathcal{R}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of words $w \in A^*$ consisting of $n_j$ copies of $j$. Define $\mathcal{D}_{wt}(x_1^{n_1} \cdots x_k^{n_k})$ to be the set of such words for which all levels $\ell_i$ are nonnegative.
The Sweep Conjecture

Conjecture: For any nonnegative integers $n_1, \ldots, n_k$ and any weight-function $wt,$

- $sw_{wt}$ maps $R(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself, and

- $sw_{wt}$ maps $D_{wt}(x_1^{n_1} \cdots x_k^{n_k})$ bijectively to itself.
# Examples of $sw_{wt}$

<table>
<thead>
<tr>
<th>Context</th>
<th>Citation</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity map</td>
<td>−1 −1</td>
<td></td>
</tr>
<tr>
<td>Reversal map</td>
<td>+1 +1</td>
<td></td>
</tr>
<tr>
<td>Dyck paths</td>
<td>L ’03</td>
<td>+1 −1</td>
</tr>
<tr>
<td>Schröder paths</td>
<td>EHKK ’03</td>
<td>+1 −1 0</td>
</tr>
<tr>
<td>Trapezoidal paths</td>
<td>L ’03</td>
<td>+1 $-m$</td>
</tr>
<tr>
<td>Square paths</td>
<td>LW ’07</td>
<td>+1 −1</td>
</tr>
<tr>
<td>Jacobians</td>
<td>GM ’13</td>
<td>$+r$ $-s$</td>
</tr>
<tr>
<td>$(a, b)$-cores</td>
<td>AHJ ’13</td>
<td>$+b$ $-a$</td>
</tr>
</tbody>
</table>
Catalan numbers

Fact: $\sum_{w \in \mathcal{D}_n} 1 = C_n = \frac{1}{n+1} \binom{2n}{n}$.

So $\sum_{w \in \mathcal{D}_3} 1 = 5 = \frac{1}{4} \binom{6}{3}$. 
**q-Catalan numbers**

Fact: \[ \sum_{w \in D_n} 1 = C_n = \frac{1}{n+1} \binom{2n}{n}. \]

So \[ \sum_{w \in D_3} q^{\text{area}(w)} = q^3 + q^2 + 2q + 1. \]
Given (G-H): Rational functions $OC_n(q, t)$ satisfying

$$OC_n(q, t) = OC_n(t, q),$$

$$OC_n(1, 1) = C_n,$$

$$OC_n(1, q) = OC_n(q, 1) = \sum_{w \in D_n} q^{\text{area}(w)}.$$

Wanted: $OC_n(q, t) = \sum_{w \in D_n} q^{\text{area}(w)} t^{\text{tstat}(w)}.$
Area = 2

Bounce = 7

Dinv = 6

Haglund

Haiman

Theorem (G-H):

\[ OC_n(q, t) = \sum_{w \in D_n} q^{\text{area}(w)} t^{\text{bounce}(w)}. \]
Symmetry of the $q, t$-Catalan

Prove combinatorially that

$$
\sum_{w \in D_n} q^{\text{area}(w)} t^{\text{dinv}(w)} = \sum_{w \in D_n} q^{\text{dinv}(w)} t^{\text{area}(w)},
$$

Or, equivalently, that

$$
\sum_{w \in D_n} q^{\text{area}(w)} t^{\text{bounce}(w)} = \sum_{w \in D_n} q^{\text{bounce}(w)} t^{\text{area}(w)}.
$$
Sweeping up statistics

Dinv: 6
Area: 2
Bounce: 7

6 → 3 → 7
2 → 6 → 3
7 → 6

Aaargh
Symmetry of the $q, t$-Catalan

Prove combinatorially that

$$\sum_{w \in D_n} q^{\text{area}(w)} t^{\text{area}(\text{sw}(w))} = \sum_{w \in D_n} q^{\text{area}(\text{sw}(w))} t^{\text{area}(w)},$$

Or, equivalently, that

$$\sum_{w \in D_n} q^{\text{area}(w)} t^{\text{area}(\text{sw}^{-1}(w))} = \sum_{w \in D_n} q^{\text{area}(\text{sw}^{-1}(w))} t^{\text{area}(w)}.$$
Slope-\((-s/r)\) \(q, t\)-Catalan

For \(r, s \in \mathbb{Z}\) and \(a, b \geq 0\), define

\[
C_{r,s,a,b}(q, t) = \sum_{w \in \mathcal{D}_{r,s}(N^a E^b)} q^{\text{area}(w)} t^{\text{area}(sw_{r,s}(w))}.
\]

Conjecture: \(C_{r,s,a,b}(q, t) = C_{r,s,a,b}(t, q)\).
Example: $a = b = 3, r = 2, s = -1$