Combinatorial structures associated to the nabla operator

BIRS
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Slides by: Greg Warrington, Wake Forest University

Joint with: Nick Loehr, Virginia Tech

Verbal stylings by: Jim Haglund, University of Pennsylvania

\*Responsible party for any errors
\( V(\mu) \)

Garsia-Haiman modules
Let \( \{(r_i, c_i)\}_{1 \leq i \leq n} \) be the coordinates of the boxes of a Ferrers diagram \( \mu \).

Set
\[
\Delta_\mu = \left| \begin{array}{cc} x^r_j & y_i^c_j \\ \end{array} \right|_{1 \leq i,j \leq n}.
\]

Let \( V(\mu) \) be the linear span of \( \Delta_\mu \) along with its partial derivatives of all orders.
$V(\mu)$ example

$$\mu = \begin{pmatrix} (1,0) \\ (0,0) \\ (0,1) \end{pmatrix}$$

$$\Delta_\mu = \begin{vmatrix} 1 & y_1 & x_1 \\ 1 & y_2 & x_2 \\ 1 & y_3 & x_3 \end{vmatrix}$$

$$\Delta_\mu = (y_2 x_3 - x_2 y_3) - (y_1 x_3 - y_3 x_1) + (y_1 x_2 - y_2 x_1).$$
\[ V(\mu) \text{ example} \]

\[ 1 \sim \Delta_\mu \]

\[ \partial x_i \sim \{ y_3 - y_2, y_3 - y_1, y_2 - y_1 \} \]

\[ \partial y_i \sim \{ x_3 - x_2, x_3 - x_1, x_2 - x_1 \} \]

\[ \partial x_i y_j \sim \pm1 \text{ or } 0 \text{ if } i = j \]

The \( n! \) Theorem (Haiman ’01):

If \( \mu \vdash n \), then \( \dim(V(\mu)) = n! \).
$S_n$-action on $V(\mu)$

Let $R = \mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n]$. $R$ is an $S_n$-module via the “diagonal action”:

$$\sigma x_i = x_{\sigma i} \quad \sigma y_i = y_{\sigma i}.$$ 

$V(\mu)$ is an $S_n$-submodule bigraded by total $x$ and $y$ degree.
Some Series

Set $V(\mu) = \bigoplus_{i,j \geq 0} V^{i,j}(\mu)$.

$$\text{Hilb}(V(\mu)) = \sum_{i,j \geq 0} \dim(V^{i,j}(\mu)) t^i q^j$$

$$\text{Frob}(V(\mu)) = \sum_{i,j \geq 0} t^i q^j \sum_{\lambda \vdash n} s_\lambda \text{Mult}[\chi^\lambda, V^{i,j}(\mu)]$$
$V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu$

$C = \tilde{H}$ Conjecture(Garsia-Haiman, ’93):

$\text{Frob}(V(\mu)) = \tilde{H}_\mu$.

Proved by Haiman, ’01.
$V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu$

Conjecture (Haglund, '04):

$$\tilde{H}_\mu = \sum_T q^{\text{inv}_\mu(T)} t^{\text{maj}_\mu(T)} x^T.$$  

Proved by Haglund-Haiman-Loehr, '05.
\[ V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu \]

\[ \Downarrow \]

\[ \text{Filled Ferrers diagrams} \]
Diagonal harmonics

“Diagonal harmonics”:

\[
\text{DH}_n = \left\{ f \in R : \sum_{i=1}^{n} \partial_{x_i}^h \partial_{y_i}^k f = 0, \forall h + k > 0 \right\}
\]

“Diagonal harmonic alternants”:

\[
\text{DHA}_n = \{ f \in \text{DH}_n : \sigma f = \text{sgn}(\sigma)f, \forall \sigma \in S_n \}
\]
\[ V(\mu) \subset DH_n \]

Action of \( \sum_i \partial^h_{x_i} \partial^k_{y_i} \): Over \( k \) and down \( h \)

Applying to \( \Delta_\mu \), we get zero if either

- a box leaves the first quadrant, or
- two boxes end up in the same place (\( \Delta_\mu \) is antisymmetric)
$V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu \xrightarrow{} \text{Filled Ferrers diagrams}$

$\downarrow \quad \downarrow$

$D\text{H}_n \quad \nabla$

$\uparrow \quad \uparrow$

$D\text{HA}_n$
Motivation:
Defined by F. Bergeron and Garsia to study $\mathcal{V}(\mu)$ intersections.

Definition:
$\nabla(\tilde{\mathcal{H}}_\mu) = T_\mu \tilde{\mathcal{H}}_\mu$, where $T_\mu \in \mathbb{Z}[q, t]$. 
Conjecture (Garsia-Haiman, ’96):

\[ \text{Frob}(\text{DH}_n) = \nabla(e_n) \]

Proved by Haiman, ’02.
The bottom row is a special case of the middle row.
Rational Function Expansions

Theorem (Garsia-Haiman):

$$\nabla(e_n) = \sum_{\mu \vdash n} \left( \frac{T_\mu M B_\mu \Pi_\mu}{w_\mu} \right) \tilde{H}_\mu.$$ 

Using the fact that \(\langle \tilde{H}_\mu, s_{1^n} \rangle = T_\mu\),

$$\langle \nabla(e_n), s_{1^n} \rangle = \sum_{\mu \vdash n} \frac{T_\mu^2 M B_\mu \Pi_\mu}{w_\mu} \in \mathbb{Q}(q, t)$$

This last formula defines the \(q, t\)-Catalan.
q, t-Catalan

Conjecture (Garsia-Haiman, ’92):

\[ C_n(q, t) \in \mathbb{N}[q, t] \]

Proved by Garsia-Haglund, ’01.
Garsia-Haiman showed:

\[ C_n(q, t) = C_n(t, q), \]

\[ C_n(1, 1) = \frac{1}{n + 1} \binom{2n}{n} = C_n, \]

\[ q^{\binom{n}{2}} C_n(q, 1/q) = \frac{1}{[n + 1]_q \left\{ \begin{array}{c} 2n \\ n, n \end{array} \right\}_q, \]

\[ C_n(1, q) = C_n(q, 1) = \sum_{D \in \mathcal{D}_n} q^{\text{area}(D)}. \]
area = 10
Looking for a tstat

Wanted: A “tstat” such that

\[ C_n(q, t) = \sum_{D \in \mathcal{D}_n} q^{\text{area}(D)} t^{t\text{stat}(D)}. \]

Haglund proposed “bounce” for tstat.
Bounce

\[ \text{boun} = 4 + 2 + 0 \]
Conjecture (Haglund, ’00):

\[ C_n(q, t) = \sum_{D \in \mathcal{D}_n} q^{\text{area}(D)} t^{\text{boun}(D)} \]

Proved by Garsia-Haglund, ’01.
Haiman’s “dinv”:

\[ \text{dinv} = 5 \]

Note: There exists a bijection \( \phi : \mathcal{D}_n \rightarrow \mathcal{D}_n \) taking

\[(\text{dinv}(D), \text{area}(D)) \mapsto (\text{area}(\phi(D)), \text{boun}(\phi(D))).\]
Labeling paradigm

For a symmetric function \( a \),

\[
\langle \nabla(a), s_1^n \rangle \rightarrow \text{Objects, unlabeled}
\]

\[
\langle \nabla(a), h_1^n \rangle \rightarrow \text{Objects, distinct labels}
\]

\[
\nabla(a) \rightarrow \text{Objects, repeated labels}
\]
\[ V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu \]
\[ \text{DH}_n \xrightarrow{\text{Frob}} \nabla(e_n) \]
\[ \text{DHA}_n \xrightarrow{\text{Hilb}} \langle \nabla(e_n), s_1^n \rangle \]

Note: \text{dinv} and \text{boun} have “labeled” versions
Representations

\[ V(\mu) \xrightarrow{\text{Frob}} \tilde{H}_\mu \]

\[ \text{DH}_n \xrightarrow{\text{Frob}} \nabla(e_n) \]

\[ \text{DHA}_n \xrightarrow{\text{Hilb}} \langle \nabla(e_n), s_1^n \rangle \]

\[ ? \xrightarrow{\text{Frob}} \nabla(\text{Other symmetric functions}) \]

Symmetric Functions

\[ \nabla(e_n) \xrightarrow{\text{rational function expansion}} \langle \nabla(e_n), s_1^n \rangle \]

\[ \text{DHA}_n \xrightarrow{\text{Hilb}} \langle \nabla(e_n), s_1^n \rangle \]

\[ \text{DHA}_n \xrightarrow{\text{Hilb}} \langle \nabla(e_n), s_1^n \rangle \]

Combinatorics

\[ \text{Filled Ferrers diagrams} \]

\[ \text{Labeled Dyck paths} \]

\[ \text{Unlabeled Dyck paths} \]

?
Conjecturally known nablas

\[ \nabla(e_n) \]

Haglund-Haiman-Loehr-Remmel-Ulyanov

Proof for \( \langle \nabla(e_n), s_1^n \rangle \): Garsia-Haglund-Haiman

\[ \nabla(p_n) \]

Loehr-W

Proof for \( \langle \nabla(p_n), s_1^n \rangle \): Can-Loehr

\[ \nabla(s_{\text{hook}}) \]

Loehr-W

Labeled, nested Dyck paths

\[ \nabla(s_\lambda) \]

Loehr-W

Labeled, Dyck paths with late first return

\[ \nabla(s_{\text{hook}}) \]

Loehr-W

Labeled square paths

\[ \nabla(s_{\text{hook}}) \]

Loehr-W

Labeled, nested Dyck paths
Conjecture (Loehr-W). For any partition $\lambda$,

$$\nabla(s_\lambda) = \text{sgn}(\lambda) \sum_{(\Pi, R) \in \text{LNDP}_\lambda} t^{\text{area}(\Pi, R)} q^{\text{dinv}(\Pi, R)} x_R,$$

where

- $\Pi = (\pi_0, \ldots, \pi_{\ell(\lambda')-1})$ is a tuple of Nested Dyck Paths
- $R = (r_0, \ldots, r_{\ell(\lambda')-1})$ is a tuple of Labels.
One term of $\nabla(s_{542})|_{m_{111}}$

Sign depends on number of rows crossed by rim hooks. Lengths of rim hooks determine lengths of Dyck paths.
One term of $\nabla (s_{542}) |_{m_{111}}$

$\begin{pmatrix} (\begin{array}{c} 4 \\ 0 \\ 0 \end{array}) \end{pmatrix}$

$(-1)^6 t q \begin{array}{c} \vdash \end{array}$
One term of $\nabla(s_{542})|_{m_{111}}$

$$\text{area} = 6 \cdot 1 + 4 \cdot 2$$

$$(−1)^6 t^{14} q$$
One term of $\nabla (s_{542}) \mid m_{111}$

$$(-1)^6 t^{14} q_2 x_1 x_2 \cdots x_{11}$$
One term of $\nabla(s_{542})|_{m_{111}}$

$$\text{dinv} = 6 + 6 +$$

$$(-1)^6 t^{14} q x_1 x_2 \cdots x_{11}$$
One term of $\nabla (s_{542}) |_{m_{111}}$

$dinv = 6 + 6 + 4$

$(-1)^6 t^{14} q^{16} x_1 x_2 \cdots x_{11}$
\[ \text{LNDP}_\lambda \]

\[ \text{LNDP}_\lambda = \{(\Pi, R)\} \] as before such that

- The \( i \)-th path in \( \Pi \) starts at \((i, i)\) and has length equal to that of the \( i \)-th hook from the top.

- The entries in \( i \)-th label vector in \( R \) strictly increase up columns of corresponding path.
And furthermore... 

Paths can’t

- cross
- share east edges
- pass through another’s start
- have $a > b$: $\overline{a} | \overline{b}$
Coefficient of $m_{211}$ in $\nabla(s_{22})$

$$\nabla(s_{22}) = -t^2q^2m_{31} - t^2q^2m_{22}$$

$$-t^2q^2(2 + t + q)m_{211}$$

$$-t^2q^2(3t + 3q + 3 + tq)m_{14}$$
LLT Polynomials
\[ \sum_{R: \ (\Pi,R) \in LNDP_\lambda} q^{dinv(\Pi,R)} x_R = q^{\text{adj}(\lambda) + n(\Gamma(\Pi))} \sum_{T \in SSYT_{\Gamma(\Pi)}} q^{dinv(T)} x_T \]