Additional Examples of Chapter 9: IIR Digital Filter Design

Example E9.1: The peak passband ripple and the minimum stopband attenuation in dB of a digital filter are \( \alpha_p = 0.15 \text{ dB} \) and \( \alpha_s = 41 \text{ dB} \). Determine the corresponding peak passband and stopband ripple values \( \delta_p \) and \( \delta_s \).

**Answer:** \( \delta_p = 1 - 10^{-\alpha_p/10} \) and \( \delta_s = 10^{-\alpha_s/10} \). Hence \( \delta_p = 1 - 10^{-0.15/20} = 0.017121127 \) and \( \delta_s = 10^{-41/20} = 0.0089125 \).

Example E9.2: Determine the peak passband ripple \( \alpha_p \) and the minimum stopband attenuation \( \alpha_s \) in dB of a digital filter with peak passband ripple \( \delta_p = 0.035 \) and peak stopband ripple \( \delta_s = 0.023 \).

**Answer:** \( \alpha_p = -20 \log_{10}(1 - \delta_p) \) and \( \alpha_s = -20 \log_{10}(\delta_s) \). Hence, \( \alpha_p = -20 \log_{10}(1 - 0.035) = 0.3094537 \text{ dB} \) and \( \alpha_s = -20 \log_{10}(0.023) = 32.76544 \text{ dB} \).

Example E9.3: Determine the digital transfer function obtained by transforming the causal analog transfer function \( H_a(s) = \frac{16(s + 2)}{(s + 3)(s^2 + 2s + 5)} \) using the impulse invariance method. Assume \( T = 0.2 \) sec.

**Answer:** Applying partial-fraction expansion we can express

\[
H_a(s) = 16 \left[ \frac{-1/8}{s + 3} + \frac{0.0625 - j0.1875}{s + 1 - j2} + \frac{0.0625 + j0.1875}{s + 1 + j2} \right] = 16 \left[ \frac{-1/8}{s + 3} + \frac{1/8s + 7/8}{s^2 + 2s + 5} \right] = \frac{-2}{s + 3} + \frac{2s + 14}{(s + 1)^2 + 2^2} = \frac{-2}{s + 3} + \frac{2(s + 1)}{(s + 1)^2 + 2^2} \times \frac{6 \times 2}{(s + 1)^2 + 2^2}.
\]

Using the results of Problems 9.7, 9.8 and 9.9 we thus arrive at

\[
G(z) = -\frac{2}{1 - e^{-3T}z^{-1}} + \frac{2(z^2 - z^{-0.4} \cos(2T))}{z^2 - 2ze^{-2T} \cos(2T) + e^{-4T}} + \frac{6ze^{-2T} \sin(2T)}{z^2 - 2ze^{-2T} \cos(2T) + e^{-4T}}. \quad \text{For } T = 0.2, \text{ we then get}
\]

\[
G(z) = -\frac{2}{1 - e^{-0.6}z^{-1}} + \frac{2(z^2 - ze^{-0.4} \cos(0.4))}{z^2 - 2ze^{-0.4} \cos(0.4) + e^{-0.8}} + \frac{6ze^{-0.4} \sin(0.4)}{z^2 - 2ze^{-0.4} \cos(0.4) + e^{-0.8}} = \frac{2(z^2 - 0.6174z)}{1 - 0.5488z^{-1} + \frac{2}{z^2 - 1.2348z + 0.4493}} + \frac{1.5662z}{z^2 - 1.2348z + 0.4493}.
\]
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\[ G(z) = \frac{2z}{z - e^{-0.9}} + \frac{3z}{z - e^{-1.2}} \]
was designed using the impulse invariance method with \( T = 2 \). Determine the parent analog transfer function.

**Answer:** Comparing \( G(z) \) with Eq. (9.59) we can write
\[ G(z) = \frac{2}{1 - e^{-0.9} z^{-1}} + \frac{3}{1 - e^{-1.2} z^{-1}} = \frac{2}{1 - e^{-\alpha T} z^{-1}} + \frac{3}{1 - e^{-\beta T} z^{-1}}. \]
Hence, \( \alpha = 3 \) and \( \beta = 4 \).
Therefore, \( H_a(s) = \frac{2}{s + 3} + \frac{3}{s + 4} \).

**Example E9.5:** The causal IIR digital transfer function
\[ G(z) = \frac{5z^2 + 4z - 1}{8z^2 + 4z} \]
was designed using the bilinear transformation method with \( T = 2 \). Determine the parent analog transfer function.

**Answer:** \( H_a(s) = G(z) \bigg|_{z = \frac{1+s}{1-s}} = \frac{5\left(\frac{1+s}{1-s}\right)^2}{8\left(\frac{1+s}{1-s}\right)^2 + 4\left(\frac{1+s}{1-s}\right)} - 1 = \frac{2 + 3s}{s^2 + 4s + 3} \).

**Example E9.6:** A lowpass IIR digital transfer function is to be designed by transforming a lowpass analog filter with a passband edge \( F_p \) at 0.5 kHz using the impulse invariance method with \( T = 0.5 \) ms. What is the normalized passband edge angular frequency \( \omega_p \) of the digital filter if the effect of aliasing is negligible? What is the normalized passband edge angular frequency \( \omega_p \) of the digital filter if it is designed using the bilinear transformation method with \( T = 0.5 \) ms?

**Answer:** For the impulse invariance design \( \omega_p = \Omega_p T = 2\pi \times 0.5 \times 10^3 \times 0.5 \times 10^{-3} = 0.5\pi \).
For the bilinear transformation method design \( \omega_p = 2\tan^{-1}\left(\frac{\Omega_p T}{2}\right) \)
\[ = 2\tan^{-1}\left(\pi F_p T\right) = 2\tan^{-1}(0.25\pi) = 0.4238447331\pi. \]
Example E9.7: A lowpass IIR digital filter has a normalized passband edge at $\omega_p = 0.3\pi$. What is the passband edge frequency in Hz of the prototype analog lowpass filter if the digital filter has been designed using the impulse invariance method with $T = 0.1$ ms? What is the passband edge frequency in Hz of the prototype analog lowpass filter if the digital filter has been designed using the bilinear transformation method with $T = 0.1$ ms?

**Answer:** For the impulse invariance design $2\pi F_p = \frac{\omega_p}{T} = 0.3\pi \times 10^{-4}$ or $F_p = 1.5$ kHz. For the bilinear transformation method design $F_p = 10^4 \tan(0.15\pi) / \pi = 1.62186$ kHz.

Example E9.8: The transfer function of a second-order lowpass IIR digital filter with a 3-dB cutoff frequency at $\omega_c = 0.42\pi$ is

$$G_{LP}(z) = \frac{0.223(1+z^{-1})^2}{1 - 0.2952z^{-1} + 0.187z^{-2}}.$$  

Design a second-order lowpass filter $H_{LP}(z)$ with a 3-dB cutoff frequency at $\phi_c = 0.57\pi$ by transforming $G_{LP}(z)$ using a lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the two lowpass filters on the same figure.

**Answer:** For $\omega_c = 0.42\pi$ and $\phi_c = 0.57\pi$, we have

$$\alpha = \frac{\sin(\omega_c - \phi_c)}{\sin(\omega_c + \phi_c)} = \frac{\sin(0.075\pi)}{\sin(0.495\pi)} = -0.233474.$$  

Thus,  

$$H_{LP}(z) = G_{LP}(z)\bigg|_{z^{-1}} = \frac{2^{-1} - \alpha}{1 - \alpha 2^{-1}} = \frac{0.223\left(1 + \frac{2^{-1} - \alpha}{1 - \alpha 2^{-1}}\right)^2}{1 - 0.2952\left(\frac{2^{-1} - \alpha}{1 - \alpha 2^{-1}}\right) + 0.187\left(\frac{2^{-1} - \alpha}{1 - \alpha 2^{-1}}\right)^2}$$

$$= \frac{0.360454(1 + 2^{1-1})^2}{1 + 2581362^{-1} + 0.18335682^{-2}}.$$
Example E9.9: Design a second-order highpass filter $H_{HP}(z)$ with a 3-dB cutoff frequency at $\omega_c = 0.61\pi$ by transforming $G_{LP}(z)$ of Example E9.8 using the lowpass-to-highpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

**Answer:** For $\omega_c = 0.42\pi$ and $\omega_c = 0.61\pi$ we have

$$\alpha = -\frac{\cos\left(\frac{\omega_c + \omega_c}{2}\right)}{\cos\left(\frac{\omega_c - \omega_c}{2}\right)} = -\frac{\cos(0.515\pi)}{\cos(-0.95\pi)} = 0.0492852.$$ 

$$H_{HP}(z) = G_{LP}(z) \bigg|_{z^{-1}} = 2^{-1} + \frac{\alpha}{1 + \alpha 2^{-1}} = \frac{0.223 \left(1 - \frac{2^{-1} + \alpha}{1 + \alpha 2^{-1}}\right)^2}{1 + 0.2952 \left(\frac{2^{-1} + \alpha}{1 + \alpha 2^{-1}}\right) + 0.187 \left(\frac{2^{-1} + \alpha}{1 + \alpha 2^{-1}}\right)^2} = \frac{0.19858(1 - 2^{-1})^2}{1 + 0.4068165 2^{-1} + 0.200963 2^{-2}}.$$
Example E9.10: The transfer function of a second-order lowpass Type 1 Chebyshev IIR digital filter with a 0.5-dB cutoff frequency at $\omega_c = 0.27\pi$ is

$$GLP(z) = \frac{0.1494(1 + z^{-1})^2}{1 - 0.7076 z^{-1} + 0.3407 z^{-2}}.$$ 

Design a fourth-order bandpass filter $H_{BP}(z)$ with a center frequency at $\omega_o = 0.45\pi$ by transforming $GLP(z)$ using the lowpass-to-bandpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

**Answer:** Since the passband edge frequencies are not specified, we use the mapping of Eq. (9.44) to map $\omega = 0$ point of the lowpass filter $GLP(z)$ to the specified center frequency $\omega_o = 0.45\pi$ of the desired bandpass filter $H_{BP}(z)$. From Eq. (9.46) we get $\lambda = \cos(\omega_o) = 0.1564347$. Substituting this value of in Eq. (9.44) we get the desired lowpass-to-bandpass transformation as

$$z^{-1} \rightarrow -z^{-1} \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} = \frac{0.1564347 z^{-1} - z^{-2}}{1 - 0.1564347 z^{-1}}.$$ 

Then, $H_{BP}(z) = GLP(z) \bigg|_{z^{-1} \rightarrow 0.1564347 z^{-1} - z^{-2}}^{1 - 0.1564347 z^{-1}} = \frac{0.1494(1 - z^{-2})^2}{1 - 0.423562 z^{-1} + 0.757725 z^{-2} - 0.217287 z^{-3} + 0.3407 z^{-4}}.$

![Gain response graph](image)

Example E9.11: A third-order Type 1 Chebyshev highpass filter with a passband edge at $\omega_p = 0.6\pi$ has a transfer function

$$G_{HP}(z) = \frac{0.0916(1 - 3z^{-1} + 3z^{-2} - z^{-3})}{1 + 0.7601 z^{-1} + 0.7021 z^{-2} + 0.2088 z^{-3}}.$$ 

Design a highpass filter with a passband edge at $\omega_p = 0.5\pi$ by transforming using the lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.
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**Answer:** \( \omega_p = 0.6\pi \), and \( \vartheta_p = 0.5\pi \). Thus, \( \alpha = \frac{\sin\left( \frac{\omega_p - \vartheta_p}{2} \right)}{\sin\left( \frac{\omega_p + \vartheta_p}{2} \right)} = \frac{\sin(0.05\pi)}{\sin(0.55\pi)} = 0.15838444. \)

Therefore, \( H_{HP}(z) = G_{HP}(z) \bigg|_{z^{-1} \rightarrow z^{-1} - 0.15838444} = \frac{0.15883792(1 - z^{-1})^3}{1 + 0.126733z^{-1} + 0.523847z^{-2} + 0.125712z^{-3}} \)

**Example E9.12:** The transfer function of a second-order notch filter with a notch frequency at 60 Hz and operating at a sampling rate of 400 Hz is

\[
G_{BS}(z) = \frac{0.954965 - 1.1226287z^{-1} + 0.954965z^{-2}}{1 - 1.1226287z^{-1} + 0.90993z^{-2}}
\]

Design a second-order notch filter \( H_{BS}(z) \) with a notch frequency at 100 Hz by transforming \( G_{BS}(z) \) using the lowpass-to-lowpass spectral transformation. Using MATLAB plot the gain responses of the both filters on the same figure.

**Answer:** The above notch filter has notch frequency at \( \omega_o = 2\pi \left( \frac{60}{400} \right) = 0.3\pi \). The desired notch frequency of the transformed filter is \( \vartheta_o = 2\pi \left( \frac{100}{400} \right) = 0.5\pi \). The lowpass-to-lowpass transformation to be used is thus given by \( z^{-1} \rightarrow \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \), where \( \lambda = \frac{\sin\left( \frac{\omega_o - \vartheta_o}{2} \right)}{\sin\left( \frac{\omega_o + \vartheta_o}{2} \right)} = \frac{\sin(0.05\pi)}{\sin(0.55\pi)} = -0.32492 \). The desired transfer function is thus given by \( H_{BS}(z) = G_{BS}(z) \bigg|_{z^{-1} \rightarrow \frac{z^{-1} - \alpha}{1 - \alpha z^{-1}}} \).
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\[
0.954965 - 1.1226287 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right) + 0.954965 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right)^2 \\
= \\
\frac{1 - 1.1226287 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right) + 0.90993 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right)^2}{1 - 1.1226287 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right) + 0.90993 \left( \frac{z^{-1} - \lambda}{1 - \lambda z^{-1}} \right)^2} \\
= \\
\frac{0.9449 - 0.1979 \times 10^{-7} z^{-1} + 0.9449 z^{-2}}{1 - 0.1979 \times 10^{-7} z^{-1} + 0.8898 z^{-2}}.
\]