Complementary Transfer Functions

• A set of digital transfer functions with complementary characteristics often finds useful applications in practice
• Four useful complementary relations are described next along with some applications

Delay Complementary Transfer Functions

• A set of \( L \) transfer functions \( \{H_i(z)\} \), \( 0 \leq i \leq L-1 \), is defined to be delay-complementary of each other if the sum of their transfer functions is equal to some integer multiple of unit delays, i.e.,
\[
\sum_{i=0}^{L-1} H_i(z) = \beta z^{-n_0}, \quad \beta \neq 0
\]
where \( n_0 \) is a nonnegative integer.

Delay Complementary Transfer Functions

• A delay-complementary pair \( \{H_0(z), H_1(z)\} \) can be readily designed if one of the pairs is a known Type 1 FIR transfer function of odd length
• Let \( H_0(z) \) be a Type 1 FIR transfer function of length \( M = 2K+1 \)
• Then its delay-complementary transfer function is given by
\[
H_1(z) = z^{-K} - H_0(z)
\]

Delay Complementary Transfer Functions

• Its delay-complementary transfer function \( H_1(z) \) has a frequency response given by
\[
H_1(e^{j\omega}) = e^{-jK\omega} H_1(\omega) = e^{-jK\omega}[1 - H_0(\omega)]
\]
• Now, in the passband, \( 1 - \delta_p \leq H_0(\omega) \leq 1 + \delta_p \), and in the stopband, \( -\delta_s \leq H_0(\omega) \leq \delta_s \)
• It follows from the above equation that in the stopband, \( -\delta_p \leq H_1(\omega) \leq \delta_p \), and in the passband, \( 1 - \delta_s \leq H_1(\omega) \leq 1 + \delta_s \)

Delay Complementary Transfer Functions

• As a result, \( H_1(z) \) has a complementary magnitude response characteristic to that of \( H_0(z) \) with a stopband exactly identical to the passband of \( H_0(z) \), and a passband that is exactly identical to the stopband of \( H_0(z) \)
• Thus, if \( H_0(z) \) is a lowpass filter, \( H_1(z) \) will be a highpass filter, and vice versa
Delay Complementary Transfer Functions

• The frequency $\omega_c$ at which $H_0(\omega_c) = H_1(\omega_c) = 0.5$ the gain responses of both filters are 6 dB below their maximum values
• The frequency $\omega_c$ is thus called the 6-dB crossover frequency

Allpass Complementary Transfer Functions

• A set of $M$ digital transfer functions, \{\hspace{1pt}H_i(z)\hspace{1pt}\} \hspace{1pt}, \hspace{1pt} 0 \leq i \leq M-1\hspace{1pt},\hspace{1pt} is defined to be allpass-complementary of each other, if the sum of their transfer functions is equal to an allpass function, i.e.,
  \[
  \sum_{i=0}^{M-1} H_i(z) = A(z)
  \]

Power-Complementary Transfer Functions

• A set of $M$ digital transfer functions, \{\hspace{1pt}H_i(z)\hspace{1pt}\} \hspace{1pt}, \hspace{1pt} 0 \leq i \leq M-1\hspace{1pt},\hspace{1pt} is defined to be power-complementary of each other, if the sum of their square-magnitude responses is equal to a constant $K$ for all values of $\omega$, i.e.,
  \[
  \sum_{i=0}^{M-1} |H_i(e^{j\omega})|^2 = K, \hspace{1pt} \text{for all} \hspace{1pt} \omega
  \]

Example

- Consider the Type 1 bandstop transfer function
  \[
  H_{BS}(z) = \frac{1}{64}(1+z^{-2})^2(1-4z^{-2}+5z^{-4}+5z^{-8}+4z^{-10}+z^{-12})
  \]
  • Its delay-complementary Type 1 bandpass transfer function is given by
    \[
    H_{BP}(z) = z^{-10} - H_{BS}(z)
    \]
    \[
    = \frac{1}{64}(1-z^{-2})^2(1+4z^{-2}+5z^{-4}+5z^{-8}+4z^{-10}+z^{-12})
    \]

Plots of the magnitude responses of $H_{BS}(z)$ and $H_{BP}(z)$ are shown below.
Power-Complementary Transfer Functions

- For a pair of power-complementary transfer functions, \(H_0(z)\) and \(H_1(z)\), the frequency \(\omega_c\) where \(|H_0(e^{j\omega_c})|^2 = |H_1(e^{j\omega_c})|^2 = 0.5\), is called the cross-over frequency
- At this frequency the gain responses of both filters are 3-dB below their maximum values
- As a result, \(\omega_c\) is called the 3-dB cross-over frequency

Example - Consider the two transfer functions
\[H_0(z) = \frac{1}{2}[A_0(z) + A_1(z)]\]
\[H_1(z) = \frac{1}{2}[A_0(z) - A_1(z)]\]
where \(A_0(z)\) and \(A_1(z)\) are stable allpass transfer functions
- Note that \(H_0(z) + H_1(z) = A_0(z)\)
- Hence, \(H_0(z)\) and \(H_1(z)\) are allpass complementary

Power-Complementary Transfer Functions

- It can be shown that \(H_0(z)\) and \(H_1(z)\) are also power-complementary
- Moreover, \(H_0(z)\) and \(H_1(z)\) are bounded-real transfer functions

Doubly-Complementary Transfer Functions

- A set of \(M\) transfer functions satisfying both the allpass complementary and the power-complementary properties is known as a doubly-complementary set

Example - The first-order lowpass transfer function
\[H_{LP}(z) = \frac{1-z^{-1}}{2\left(\frac{1+z^{-1}}{1-\alpha z^{-1}}\right)}\]
can be expressed as
\[H_{LP}(z) = \frac{1}{2}\left(\frac{1+2\alpha+z^{-1}}{1-\alpha z^{-1}}\right) = \frac{1}{2}[A_0(z) + A_1(z)]\]
where \(A_0(z) = 1\), \(A_0(z) = -\frac{\alpha + z^{-1}}{1-\alpha z^{-1}}\)
Doubly-Complementary Transfer Functions

• Its power-complementary highpass transfer function is thus given by
\[ H_{HP}(z) = \frac{1}{2} [\mathbf{A}(z) - \mathbf{A}(z)] = \frac{1}{2} \left( 1 - \frac{\alpha + z^{-1}}{1 - \alpha z^{-1}} \right) \]

• The above expression is precisely the first-order highpass transfer function described earlier.

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Power-Symmetric Filters

• A real-coefficient causal digital filter with a transfer function \( H(z) \) is said to be a power-symmetric filter if it satisfies the condition
\[ H(z)H(z^{-1}) + H(-z)H(-z^{-1}) = K \]
where \( K > 0 \) is a constant.

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Power-Symmetric Filters

• It can be shown that the gain function \( G(\omega) \) of a power-symmetric transfer function at \( \omega = \pi \) is given by
\[ 10 \log_{10} K - 3 \text{ dB} \]

• If we define \( G(z) = H(-z) \), then it follows from the definition of the power-symmetric filter that \( H(z) \) and \( G(z) \) are power-complementary as
\[ H(z)H(z^{-1}) + G(z)G(z^{-1}) = \text{a constant} \]

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Conjugate Quadratic Filters

• If a power-symmetric filter has an FIR transfer function \( H(z) \) of order \( N \), then the FIR digital filter with a transfer function
\[ G(z) = z^{-N} H(-z^{-1}) \]
is called a conjugate quadratic filter of \( H(z) \) and vice-versa.
Conjugate Quadratic Filters

• It follows from the definition that \( G(z) \) is also a power-symmetric causal filter.
• It also can be seen that a pair of conjugate quadratic filters \( H(z) \) and \( G(z) \) are also power-complementary.