**Time-Domain Characterization of LTI Discrete-Time System**

• Input-Output Relationship -
  A consequence of the linear, time-invariance property is that an LTI discrete-time system is completely characterized by its impulse response

  ➡ Knowing the impulse response one can compute the output of the system for any arbitrary input

• Let \( h[n] \) denote the impulse response of a LTI discrete-time system

• We compute its output \( y[n] \) for the input:

\[
x[n] = 0.5\delta[n + 2] + 1.5\delta[n - 1] - \delta[n - 2] + 0.75\delta[n - 5]
\]

• As the system is linear, we can compute its outputs for each member of the input separately and add the individual outputs to determine \( y[n] \)

• Likewise, as the system is linear

\[
y[n] = 0.5h[n + 2] + 1.5h[n - 1] - h[n - 2] + 0.75h[n - 5]
\]

• Hence because of the linearity property we get

\[
y[n] = 0.5h[n + 2] + 1.5h[n - 1] - h[n - 2] + 0.75h[n - 5]
\]

• Now, any arbitrary input sequence \( x[n] \) can be expressed as a linear combination of delayed and advanced unit sample sequences in the form

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]
\]

• The response of the LTI system to an input \( x[k] \delta[n - k] \) will be \( x[k] h[n - k] \)

• Hence, the response \( y[n] \) to an input

\[
x[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k]
\]

is given by

\[
y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]
\]

which can be alternately written as

\[
y[n] = \sum_{k=-\infty}^{\infty} x[n - k] h[k]
\]
Convolution Sum

- The summation
  \[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k] h[n] \]
  is thus the convolution sum of the sequences \( x[n] \) and \( h[n] \) and represented compactly as
  \[ y[n] = x[n] \bigcirc h[n] \]

Example – Consider an LTI discrete-time system with an impulse response \( h[n] \) generating an output \( y[n] \) for an input \( x[n] \):
  \[ y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-k] = x[n] \bigcirc h[n] \]
- We determine the output \( y[n] \) of an LTI discrete-time system with an impulse response \( h[n-m] \) for the same input \( x[n] \)

Now
  \[ y_1[n] = \sum_{k=-\infty}^{\infty} x[k] h[n-m-k] = x[n] \bigcirc h[n-m] \]
- Hence,
  \[ y_1[n] = y[n-m] \]

Properties -
- Commutative property:
  \[ x[n] \bigcirc h[n] = h[n] \bigcirc x[n] \]
- Associative property:
  \[ (x[n] \bigcirc h[n]) \bigcirc y[n] = x[n] \bigcirc (h[n] \bigcirc y[n]) \]
- Distributive property:
  \[ x[n] \bigcirc (h[n] + y[n]) = x[n] \bigcirc h[n] + x[n] \bigcirc y[n] \]

Interpretation -
- 1) Time-reverse \( h[k] \) to form \( h[-k] \)
- 2) Shift \( h[-k] \) to the right by \( n \) sampling periods if \( n > 0 \) or shift to the left by \( n \) sampling periods if \( n < 0 \) to form \( h[n-k] \)
- 3) Form the product \( v[k] = x[k] h[n-k] \)
- 4) Sum all samples of \( v[k] \) to develop the \( n \)-th sample of \( y[n] \) of the convolution sum

Schematic Representation -

- The computation of an output sample using the convolution sum is simply a sum of products
- Involves fairly simple operations such as additions, multiplications, and delays
Time-Domain Characterization of LTI Discrete-Time System

• In practice, if either the input or the impulse response is of finite length, the convolution sum can be used to compute the output sample as it involves a finite sum of products.
• If both the input sequence and the impulse response sequence are of finite length, the output sequence is also of finite length.

Tabular Method of Convolution Sum Computation

• Can be used to convolve two finite-length sequences.
• Consider the convolution of \{g[n]\}, 0 ≤ n ≤ 3, with \{h[n]\}, 0 ≤ n ≤ 2, generating the sequence \( y[n] = g[n] \ast h[n] \).
• Samples of \{g[n]\} and \{h[n]\} are then multiplied using the conventional multiplication method without any carry operation.

Tabular Method of Convolution Sum Computation

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>g[n]</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>h[n]</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The samples \( y[n] \) generated by the convolution sum are obtained by adding the entries in the column above each sample.

Tabular Method of Convolution Sum Computation

• The samples \( y[n] \) are given by:
  \( y[0] = g[0]h[0] \)
  \( y[1] = g[1]h[0] + g[0]h[1] \)

Tabular Method of Convolution Sum Computation

• The method can also be applied to convolve two finite-length two-sided sequences.
• In this case, a decimal point is first placed to the right of the sample with the time index \( n = 0 \) for each sequence.
• Next, convolution is computed ignoring the location of the decimal point.
Tabular Method of Convolution Sum Computation

- Finally, the decimal point is inserted according to the rules of conventional multiplication.
- The sample immediately to the left of the decimal point is then located at the time index \( n = 0 \).

Convolution Using MATLAB

- The M-file `conv` implements the convolution sum of two finite-length sequences.
- If \( a = [-2 0 1 -1 3] \)
  \( b = [1 2 0 -1] \)
then \( \text{conv}(a, b) \) yields \([-2 -4 1 3 15 1 -3]\).

Stability Condition of an LTI Discrete-Time System

- **BIBO Stability Condition** - A discrete-time is BIBO stable if and only if the output sequence \( \{y[n]\} \) remains bounded for all bounded input sequence \( \{x[n]\} \).
- An LTI discrete-time system is BIBO stable if and only if its impulse response sequence \( \{h[n]\} \) is absolutely summable, i.e.
  \[
  S = \sum_{n=-\infty}^{\infty} |h[n]| < \infty
  \]

Proof: Assume \( h[n] \) is a real sequence.
- Since the input sequence \( x[n] \) is bounded we have
  \[
  |x[n]| \leq B_x < \infty
  \]
- Therefore
  \[
  |y[n]| = \sum_{k=-\infty}^{\infty} h[k]x[n-k] \leq \sum_{k=-\infty}^{\infty} |h[k]| |x[n-k]|
  \]
  \[
  \leq B_x \sum_{k=-\infty}^{\infty} |h[k]| = B_x S
  \]

Stability Condition of an LTI Discrete-Time System

- Thus, \( S < \infty \) implies \( |y[n]| \leq B_y < \infty \), indicating that \( y[n] \) is also bounded.
- To prove the converse, assume \( y[n] \) is bounded, i.e., \( |y[n]| \leq B_y \).
- Consider the input given by
  \[
  x[n] = \begin{cases} 
  \text{sgn}(h[-n]), & \text{if } h[-n] \neq 0 \\
  K, & \text{if } h[-n] = 0
  \end{cases}
  \]
where \( \text{sgn}(c) = +1 \) if \( c > 0 \) and \( \text{sgn}(c) = -1 \) if \( c < 0 \) and \( |K| \leq 1 \).
- Note: Since \( |x[n]| \leq 1 \), \( \{x[n]\} \) is obviously bounded.
- For this input, \( y[n] \) at \( n = 0 \) is
  \[
  y[0] = \sum_{k=-\infty}^{\infty} \text{sgn}(h[k])h[k] = S \leq B_y < \infty
  \]
- Therefore, \( |y[n]| \leq B_y \) implies \( S < \infty \).
Stability Condition of an LTI Discrete-Time System

- **Example** - Consider an LTI discrete-time system with an impulse response \( h[n] = (\alpha)^n \mu[n] \)
  - For this system, \( S = \sum_{n=-\infty}^{\infty} |x^n[n]| = \sum_{n=0}^{\infty} |x^n| = \frac{1}{1-|\alpha|} \) if \( |\alpha| < 1 \)
  - Therefore \( S < \infty \) if \( |\alpha| < 1 \) for which the system is BIBO stable
  - If \( |\alpha| = 1 \), the system is not BIBO stable

Causality Condition of an LTI Discrete-Time System

- Let \( x_1[n] \) and \( x_2[n] \) be two input sequences with
  \( x_1[n] = x_2[n] \) for \( n \leq n_0 \)
  \( x_1[n] \neq x_2[n] \) for \( n > n_0 \)
- The corresponding output samples at \( n = n_0 \) of an LTI system with an impulse response \( \{ h[n] \} \) are then given by

**Example** - Consider the discrete-time system defined by
\[
\sum_{k=-\infty}^{\infty} h[k] x_1[n_0 - k] = \sum_{k=0}^{\infty} h[k] x_1[n_0 - k] + \sum_{k=-\infty}^{\infty} h[k] x_2[n_0 - k]
\]
\[
y_1[n_0] = \sum_{k=-\infty}^{\infty} h[k] x_1[n_0 - k] + \sum_{k=-\infty}^{\infty} h[k] x_2[n_0 - k]
\]
\[
y_2[n_0] = \sum_{k=0}^{\infty} h[k] x_2[n_0 - k]
\]

An LTI discrete-time system is **causal** if and only if its impulse response \( \{ h[n] \} \) is a causal sequence

**Example** - The discrete-time system defined by
\[
y[n] = \alpha_1 x[n] + \alpha_2 x[n-1] + \alpha_3 x[n-2] + \alpha_4 x[n-3]
\]
is a causal system as it has a causal impulse response \( \{ h[n] \} \) that is causal.
Causality Condition of an LTI Discrete-Time System

- **Example** - The discrete-time accumulator defined by

  \[ y[n] = \sum_{\ell=-\infty}^{\ell(n)} \delta[\ell] = \mu[n] \]

  is a causal system as it has a causal impulse response given by

  \[ h[n] = \sum_{\ell=\infty}^{\ell(n)} \delta[\ell] = \mu[n] \]

- **Note**: A noncausal LTI discrete-time system with a finite-length impulse response can often be realized as a causal system by inserting an appropriate amount of delay.

- For example, a causal version of the factor-of-2 interpolator is obtained by delaying the input by one sample period:

  \[ y[n] = x_u[n-1] + \frac{1}{2}(x_u[n-2] + x_u[n]) \]

Simple Interconnection Schemes

- Two simple interconnection schemes are:
  - Cascade Connection
  - Parallel Connection

Cascade Connection

- Impulse response \( h[n] \) of the cascade of two LTI discrete-time systems with impulse responses \( h_1[n] \) and \( h_2[n] \) is given by

  \[ h[n] = h_1[n] \oplus h_2[n] \]

- **Note**: The ordering of the systems in the cascade has no effect on the overall impulse response because of the commutative property of convolution.

  - A cascade connection of two stable systems is stable.
  - A cascade connection of two passive (lossless) systems is passive (lossless).
Cascade Connection

- An application is in the development of an inverse system
- If the cascade connection satisfies the relation
  \[ h_1[n] \oplus h_2[n] = \delta[n] \]
  then the LTI system \( h_1[n] \) is said to be the inverse of \( h_2[n] \) and vice-versa

Example - Consider the discrete-time accumulator with an impulse response \( \mu[n] \)
- Its inverse system satisfy the condition
  \[ \mu[n] \oplus h_2[n] = \delta[n] \]
- It follows from the above that \( h_2[n] = 0 \) for \( n < 0 \) and
  \[ h_2[0] = 1 \]
  \[ \sum_{\ell=0}^{n} h_2[\ell] = 0 \] for \( n \geq 1 \)

Parallel Connection

- Impulse response \( h[n] \) of the parallel connection of two LTI discrete-time systems with impulse responses \( h_1[n] \) and \( h_2[n] \) is given by
  \[ h[n] = h_1[n] + h_2[n] \]

Simple Interconnection Schemes

- Consider the discrete-time system where
  \[ h_1[n] = \delta[n] + 0.5\delta[n-1], \]
  \[ h_2[n] = 0.5\delta[n] - 0.25\delta[n-1], \]
  \[ h_3[n] = 2\delta[n], \]
  \[ h_4[n] = -2(0.5)^n \mu[n] \]
Simple Interconnection Schemes

- Simplifying the block-diagram we obtain

\[
\begin{align*}
\sum_{n=0}^{\infty} h_2[n] & = h_2[n] + h_3[n] + h_4[n] \\
\sum_{n=0}^{\infty} h_3[n] & = h_3[n] + h_4[n] \\
\sum_{n=0}^{\infty} h_4[n] & = h_4[n]
\end{align*}
\]

- Overall impulse response \( h[n] \) is given by

\[
\begin{align*}
h[n] & = h_1[n] + h_2[n] \otimes (h_3[n] + h_4[n]) \\
& = h_1[n] + h_2[n] \otimes h_3[n] + h_2[n] \otimes h_4[n]
\end{align*}
\]

- Now,

\[
\begin{align*}
h_2[n] \otimes h_3[n] & = \left(\frac{1}{2} \delta[n] - \frac{1}{2} \delta[n-1]\right) \otimes 2\delta[n] \\
& = \delta[n] - \frac{1}{2} \delta[n-1]
\end{align*}
\]

Simple Interconnection Schemes

\[
\begin{align*}
h_2[n] \otimes h_4[n] & = \left(\frac{1}{2} \delta[n] - \frac{1}{4} \delta[n-1]\right) \otimes \left(-2\left(\frac{1}{2}\right)^n \mu[n]\right) \\
& = -\left(\frac{1}{2}\right)^n \mu[n] + \frac{1}{2} \left(\frac{1}{2}\right)^{n-1} \mu[n-1] \\
& = -\left(\frac{1}{2}\right)^n \mu[n] + \frac{1}{2} \mu[n] \\
& = -\left(\frac{1}{2}\right)^n \delta[n]
\end{align*}
\]

- Therefore

\[
\begin{align*}
h[n] & = \delta[n] + \frac{1}{2} \delta[n-1] + \delta[n] - \frac{1}{2} \delta[n-1] - \delta[n] = \delta[n]
\end{align*}
\]

Finite-Dimensional LTI Discrete-Time Systems

- An important subclass of LTI discrete-time systems is characterized by a linear constant coefficient difference equation of the form

\[
\sum_{k=0}^{N} d_k y[n-k] = \sum_{k=0}^{M} p_k x[n-k]
\]

- \( x[n] \) and \( y[n] \) are, respectively, the input and the output of the system

- \( \{d_k\} \) and \( \{p_k\} \) are constants characterizing the system

- If we assume the system to be causal, then the output \( y[n] \) can be recursively computed using

\[
y[n] = -\sum_{k=1}^{N} d_k y[n-k] + \sum_{k=0}^{M} p_k x[n-k]
\]

provided \( d_0 \neq 0 \)

- \( y[n] \) can be computed for all \( n \geq n_0 \), knowing \( x[n] \) and the initial conditions

\[
y[n_0-1], y[n_0-2], ..., y[n_0-N]
\]
Classification of LTI Discrete-Time Systems

Based on Impulse Response Length -

- If the impulse response $h[n]$ is of finite length, i.e.,
  
  $h[n] = 0$ for $n < N_1$ and $n > N_2$, $N_1 < N_2$

  then it is known as a finite impulse response (FIR) discrete-time system

- The convolution sum description here is

  $y[n] = \sum_{k=N_1}^{N_2} h[k] x[n-k]$ 

Classification of LTI Discrete-Time Systems

- The output $y[n]$ of an FIR LTI discrete-time system can be computed directly from the convolution sum as it is a finite sum of products

- Examples of FIR LTI discrete-time systems are the moving-average system and the linear interpolators

Classification of LTI Discrete-Time Systems

- If the impulse response is of infinite length, then it is known as an infinite impulse response (IIR) discrete-time system

- The class of IIR systems we are concerned with in this course are characterized by linear constant coefficient difference equations

Classification of LTI Discrete-Time Systems

- Example - The discrete-time accumulator defined by

  $y[n] = y[n-1] + x[n]$

  is seen to be an IIR system

Classification of LTI Discrete-Time Systems

- Example - The familiar numerical integration formulas that are used to numerically solve integrals of the form

  $y(t) = \int_0^t x(\tau)d\tau$

  can be shown to be characterized by linear constant coefficient difference equations, and hence, are examples of IIR systems

Classification of LTI Discrete-Time Systems

- If we divide the interval of integration into $n$ equal parts of length $T$, then the previous integral can be rewritten as

  $y(nT) = y((n-1)T) + \int_{(n-1)T}^{nT} x(\tau)d\tau$

  where we have set $t = nT$ and used the notation

  $y(nT) = \int_0^{nT} x(\tau)d\tau$
Classification of LTI Discrete-Time Systems

• Using the trapezoidal method we can write
  \[
  \int_{(n-1)T}^{nT} x(\tau)d\tau = \frac{T}{2} \{x((n-1)T) + x(nT)\}
  \]

• Hence, a numerical representation of the definite integral is given by

\[
y(nT) = y((n-1)T) + \frac{T}{2} \{x((n-1)T) + x(nT)\}
\]

Classification of LTI Discrete-Time Systems

• Let \( y[n] = y(nT) \) and \( x[n] = x(nT) \)

• Then

\[
y(nT) = y((n-1)T) + \frac{T}{2} \{x((n-1)T) + x(nT)\}
\]

reduces to

\[
y[n] = y[n-1] + \frac{T}{2} \{x[n] + x[n-1]\}
\]

which is recognized as the difference equation representation of a first-order IIR discrete-time system

Classification of LTI Discrete-Time Systems

Based on the Output Calculation Process

• Nonrecursive System - Here the output can be calculated sequentially, knowing only the present and past input samples

• Recursive System - Here the output computation involves past output samples in addition to the present and past input samples

Classification of LTI Discrete-Time Systems

Based on the Coefficients -

• Real Discrete-Time System - The impulse response samples are real valued

• Complex Discrete-Time System - The impulse response samples are complex valued