THE STEINER–LEHMUS THEOREM

by Roger Cooke

Introduction. In 1840, a schoolteacher in Berlin named Daniel Christian Ludolph Lehmus (1780–1863) wrote to Jacques Charles François Sturm (1803–1855), conjecturing that a triangle having two equal internal angle bisectors is isosceles. (An internal angle bisector of a triangle, as the name implies, is the portion of the bisector of one of its internal angles that lies inside the triangle.) Remarkably, this elementary fact does not appear to have been noticed earlier. Sturm publicized this conjecture, and a little later, Professor Jakob Steiner (1796–1863) at the University of Berlin produced a proof. Subsequently, Lehmus produced another proof.

This result, now called the Steiner–Lehmus Theorem, has gained notoriety as a simple and seemingly obvious fact that is rather difficult to prove. However, bright high-school students frequently produce proofs of it, so that it is not by any means an impossible task to do so. The Steiner–Lehmus Theorem has generated a large amount of literature (see the *Wikipedia* article under that name). Apparently, no one has yet produced a direct proof of this theorem. All of the dozens of proofs that exist use some form of the Law of the Excluded Middle.

A cheap analytic proof. One might think it would be a trivial matter to prove this theorem using analytic geometry. The triangle could be situated so that one vertex is at $(0,0)$, another at $(a,0)$, and the third at (b,c), where a, b, and c are arbitrary positive numbers. For a triangle with vertices at $(0,0)$, $(a,0)$, (t,c), where a and c are positive numbers and t is any real number, the square of the length of the internal bisector of the angle with vertex at $(0,0)$ is

$$f(t) = \frac{2a^2(t^2+c^2+t\sqrt{t^2+c^2})}{(a+\sqrt{t^2+c^2})^2} = 2a^2\Big(1 - \frac{a^2+(2a-t)\sqrt{t^2+c^2}}{(a+\sqrt{t^2+c^2})^2}\Big)\,.$$

By reflecting this triangle about the line $x = a/2$, one can see that the square of the length of the internal bisector of the angle with vertex at $(a,0)$ in the same triangle is $f(a-t)$.

The Steiner–Lehmus theorem therefore says that the equality $f(b) = f(a-b)$ implies $b = a-b$. Thus, a proof follows from the fact that the function $f(t)$ is monotonically increasing.

This proof, however, uses calculus. There does not seem to be any simple way to change it into a geometric proof.

A prettier synthetic proof. I first heard about this theorem about 25 years ago from a former student of mine, Skip Gates, who had become a teacher. I did a bit of work and came up with the proof given below. I suspect that, like most of my best discoveries, this is actually a rediscovery of something known long ago. But since I haven't seen any proof quite like it in my not-very-systematic searches of the literature, and since my friend Tony Trono, who collects proofs of this theorem, assures me that my proof is *sui generis* among the ones he has seen, I'm emboldened to post it here for the edification of the interested visitor to this website.

This proof, while not a direct proof of the Steiner–Lehmus Theorem, is a direct proof of a slightly stronger statement that implies the Steiner–Lehmus Theorem.

Theorem. *If two sides of a triangle are unequal, the shorter side meets the longer internal angle bisector. Equivalently, if two angles of a triangle are unequal, the smaller angle has the longer internal bisector.*

Proof: Throughout, we shall consider a triangle ABC with side AC longer than side BC ($\angle A$ smaller than $\angle B$). We shall demonstrate that the internal bisector AD is longer than the internal bisector BE. There are two cases to consider, shown in Figs. 1 and 2. In Fig. 1, $\angle AEB$ is right or obtuse; in Fig. 2, it is acute.

Case 1: Assuming that $\angle AEB$ is obtuse or right, draw angle ABF equal to $\angle A$, with side BF meeting AC at point F, as shown in Fig. 1. (This point lies between A and C, since $\angle A$ is smaller than $\angle B$. In Fig. 1, we show F between E and C. It will be between A and E if $\angle A$ is less than half of $\angle B$, but our argument remains valid in that case without any changes.) Thus $\triangle ABF$ is isosceles. Side BF intersects the internal bisector AD. Let G denote the point of intersection.

Next, draw the internal bisector BH of $\angle ABF$. Since $\angle A$ is less than $\angle B$, the point H lies between A and E.

Now, since $\triangle ABF$ is isosceles, with equal sides AF and BF, the two internal angle bisectors AG and BH are equal. We then have $AD > AG = BH > BE$, the last inequality because $\angle AEB$ is right or obtuse.

Case 2: Now assume that angle AEB is acute. (This occurs when $\angle A + \frac{1}{2}\angle B$ is larger than a right angle.) We observe that $\angle ADB = \angle AEB - \frac{1}{2}(\angle B - \angle A)$, so that $\angle ADB$ is smaller than $\angle AEB$. It follows trivially that $\angle ADB$ is also acute. Let the point where the angle bisectors AD and BE meet be labeled G. This point is the center of the circle inscribed in $\triangle ABC$. Drop (equal) perpendiculars GH and GI from G to AE and BD respectively. The feet of these perpendiculars, H and I, lie within the segments, since the triangles AGE and BGD have acute angles at the ends of these two segments. Since $GH = GI$ and $\angle HEG$ is larger than $\angle IDG$, it follows that DG is longer than EG.

But, in $\triangle ABG$, the angle GAB is smaller than GBA, so that $BG < AG$. We thus have

$$BE = BG + EG < AG + DG = AD\,.$$

The proof is now complete.

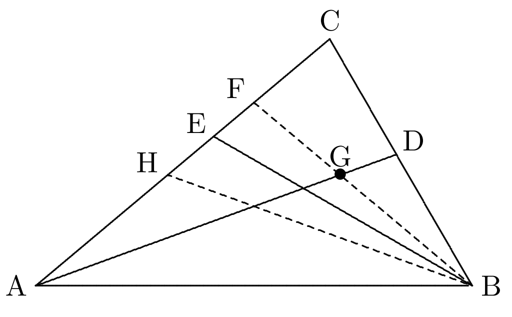

Fig. 1

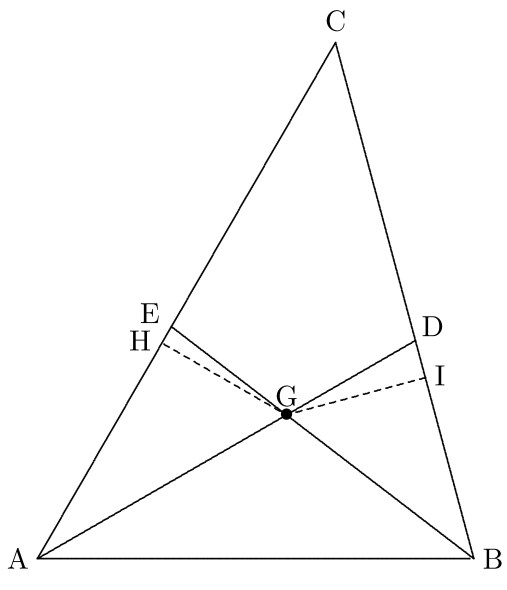

Fig. 2