THE SEVENTY-FIRST ANNUAL WILLIAM LOWELL PUTNAM EXAMINATION
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Problem A1

Given a positive integer n, what is the largest k such that the numbers 1, 2,..., n can be put into k boxes so that the sum of the numbers in each box is the same? [When $n = 8$, the example $\{1,2,3,6\}$, $\{4,8\}$, $\{5,7\}$ shows that the largest k is *at least* 3.]

Solution: The largest k is $k = \frac{n+1}{2}$ if n is odd, and $k = \frac{n}{2}$ if n is even.

In the first case, the boxes are $\{n\}$, $\{1, n-1\}$, $\{2, n-2\}$,..., $\{\frac{n-1}{2}, \frac{n+1}{2}\}$. In the second case, they are $\{1, n\}$, $\{2, n-1\}$,..., $\{\frac{n}{2}, \frac{n}{2}+1\}$.

The proof goes as follows. There can be at most one box with a single number in it, and that number must be n. Then all the other boxes must have at least 2 elements, and to get the maximal number of boxes, we put the minimal number of elements in each. This takes care of the case of n odd.

If n is even, the partition cannot contain the singleton set $\{n\}$, since the sum of the numbers in each box has to divide $\frac{n(n+1)}{2}$. Thus, the best we can do is to use only 2-element boxes, as shown here.

Problem A2

Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f'(x) = \frac{f(x+n) - f(x)}{n}$$

for all real numbers x and all positive integers n

Solution: These are precisely the polynomials of degree 1 with real coefficients, that is, $f(x) = a + bx$ for some constants a and b.

Indeed, suppose f has this property. Let $g(x) = f(x) - f(0) + \big(f(0) - f(1)\big)x$. Then

$$\frac{g(x+n) - g(x)}{n} = \frac{f(x+n) - f(x)}{n} + f(0) - f(1)\,.$$

and

$$g'(x) = f'(x) + f(0) - f(1)\,.$$

Hence $g(x)$ also has this property, and in addition, $g(0) = 0 = g(1)$. Taking $n = 1$, $x = 0$, we find that $g'(0) = 0$. It then follows that $g(n) = 0$ for all positive integers n, since $g(n) = ng'(0) + g(0)$.

It now follows that $g'(m) = 0$ for all integers m, positive or negative. Indeed, we have

$$g'(m) = \frac{g(m+n) - g(m)}{n}$$

for all integers m and all positive integers n. Now the right-hand side here equals $-g(m)/n$ for all $n > -m$, and letting $n \to \infty$, we see that $g'(m) = 0$.

But then, if $n > -m$, we have $g(m) = -ng'(m) = 0$ for all integers m, positive or negative.

Now let r be any integer, positive or negative. Then $g(x)$ has extreme values on $[r, r+1]$ that are attained at points $c \in (r, r+1)$ where $g'(c) = 0$. It then follows just as above that $g(c+n) = g(c)$

for all integers n, positive or negative. Thus, the maximum and minimum values of $g(x)$ are the same in every interval $[r, r+1]$. It follows that $g(x)$ is a bounded function. But then, letting n tend to ∞, we see that $g'(x) \equiv 0$. That is, $g(x)$ is a constant, and in fact $g(x) \equiv 0$. This means that

$$f(x) = f(0) + \big(f(1) - f(0)\big)x\,.$$

Problem A3

Suppose that the function $h : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives and satisfies the equation

$$h(x, y) = a\frac{\partial h}{\partial x}(x, y) + b\frac{\partial h}{\partial y}(x, y)$$

for some constants a, b. Prove that if there is a constant M such that $|h(x, y)| \leq M$ for all $(x, y) \in \mathbb{R}^2$, then h is identically zero.

Solution: If $a = 0 = b$, there is nothing to prove, so we shall assume that $a \neq 0$. Given any point (x, y), let $c = y - \frac{bx}{a}$. Consider the function $\varphi(t) = h(at, bt + c)$. We have

$$\varphi'(t) = a\frac{\partial h}{\partial x}(at, bt + c) + b\frac{\partial h}{\partial y}(at, bt + c) = h(at, bt + c) = \varphi(t).$$

Hence for some constant k, $\varphi(t) = ke^t$. Since $\varphi(t)$ is bounded, $k = 0$, and therefore $\varphi(t) \equiv 0$. Taking $t = \frac{x}{a}$, we find

$$0 = \varphi(t) = h(x, y)\,.$$

Problem A4

Prove that for each positive integer n, the number $10^{10^{10^n}} + 10^{10^n} + 10^n - 1$ is not prime.

Solution: Let $n = 2^r k$, where r is a nonnegative integer and k an odd positive integer. We claim that this number is divisible by $10^{2^r} + 1$. In fact,

$$10^n = 2^n \cdot 5^n = 2^{r+1}u\,,$$

where $u = 5^n 2^{2^r k - r - 1}$, and $2^r k - r - 1$ is a nonnegative integer. (If $r = 0$, it is $k - 1$. If $r = 1$, it is $2(k - 1)$, and for $r \geq 2$ it is at least $2^r - r - 1 \geq 1$.)

It now follows that

$$10^{10^{10^n}} = 2^{r+1}v$$

for some positive integer v. The number in question can now be written as

$$\big(10^{2^r}\big)^{2v} + \big(10^{2^r}\big)^{2u} + \big(10^{2^r}\big)^{k} - 1\,.$$

Since $10^{2^r} \equiv -1 \bmod 10^{2^r} + 1$, we see that, mod $10^{2^r} + 1$ we have a number equal to

$$(-1)^{2v} + (-1)^{2u} + (-1)^k - 1\,,$$

and this is zero since k is odd.

Problem A5

Let G be a group, with operation $*$. Suppose that

(i) G is a subset of $\mathbb{R}^3$ (but $*$ need not be related to addition of vectors);

(ii) For each $\mathbf{a}$, $\mathbf{b} \in G$, either $\mathbf{a}\times\mathbf{b} = \mathbf{a}*\mathbf{b}$ or $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ (or both), where $\times$ is the usual cross product in $\mathbb{R}^3$.

Prove that $\mathbf{a}\times\mathbf{b} = \mathbf{0}$ for all $\mathbf{a}$, $\mathbf{b} \in G$.

Solution: Since the cross product $\mathbf{a}\times\mathbf{b}$ is zero if and only if $\{\mathbf{a},\mathbf{b}\}$ is a linearly dependent set, what we are really trying to establish here is that the entire group G is contained in a single line through the origin, that is, every vector in G is a scalar multiple of any non-zero element of G. We do this in four steps.

Step 1: Let $\mathbf{e}$ be the identity of the group. We first establish that $\mathbf{e}\times\mathbf{a} = \mathbf{0}$ for all $\mathbf{a} \in G$. This is obvious if $\mathbf{e} = \mathbf{0}$ or $\mathbf{a} = \mathbf{0}$. Supposing that neither of these is the case, we assume that $\mathbf{e}\times\mathbf{a} \neq \mathbf{0}$. By hypothesis, then, $\mathbf{e}\times\mathbf{a} = \mathbf{e}*\mathbf{a} = \mathbf{a}$. But since $\mathbf{e}\times\mathbf{a}$ is perpendicular to $\mathbf{a}$, this means that $\mathbf{a}\cdot\mathbf{a} = 0$, and hence that $\mathbf{a} = \mathbf{0}$, contrary to hypothesis. Thus, using the principle that (not-p implies p) implies p, we conclude that $\mathbf{e}\times\mathbf{a} = \mathbf{0}$ for all $\mathbf{a} \in G$. In particular, if $\mathbf{e} \neq \mathbf{0}$, it follows that all elements of G are scalar multiples of $\mathbf{e}$, and we are done. From now on, we assume that $\mathbf{e} = \mathbf{0}$.

Step 2: Now let the inverse of $\mathbf{a}$ be denoted $\mathbf{a}^{-1}$. We next establish that $\mathbf{a} \times \mathbf{a}^{-1} = \mathbf{0}$. If this equation does not hold, then $\mathbf{a} \times \mathbf{a}^{-1} = \mathbf{a} * \mathbf{a}^{-1} = \mathbf{e} = \mathbf{0}$. Once again, using the principle that (not-p implies p) implies p, we conclude that $\mathbf{a} \times \mathbf{a}^{-1} = \mathbf{0}$ for all $\mathbf{a} \in G$.

In particular, it follows that $\mathbf{a}$ and $\mathbf{a}^{-1}$ are linearly dependent for all $\mathbf{a} \in G$, and neither of these is the zero vector if $\mathbf{a} \neq \mathbf{e}$.

Step 3: Now suppose $\mathbf{a} \in G$ and $\mathbf{b} \in G$, and $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. In particular, neither of these vectors is the identity of the group, and so $\mathbf{a}^{-1} \neq \mathbf{0}$. Then by hypothesis $\mathbf{a} \times \mathbf{b} = \mathbf{a} * \mathbf{b}$, and this vector is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$. In particular, it is perpendicular to $\mathbf{a}^{-1}$, which is a nonzero vector that is a scalar multiple of $\mathbf{a}$. Therefore,

$$\mathbf{a}^{-1} \times \big(\mathbf{a} \times \mathbf{b}\big) \neq \mathbf{0}\,,$$

and so this last vector equals

$$\mathbf{a}^{-1} * (\mathbf{a} * \mathbf{b}) = (\mathbf{a}^{-1} * \mathbf{a}) * \mathbf{b} = \mathbf{b}\,.$$

It now follows that $\mathbf{b}$ is perpendicular to $\mathbf{a}$. That is, any two elements of the group whose cross product is not zero must be perpendicular to each other. This means G is contained in the union of three mutually perpendicular lines through the origin. We would like to reduce that number to one line.

Step 4: Let us denote the portions of the group G in each of these lines by A, B, and C. Common to all three sets is the identity $\mathbf{e}$. We would like to show that two of these contain only the element $\mathbf{e}$. Let us assume that A contains a non-identity element of the group.

We claim that A, B, and C are subgroups of G, in fact, normal subgroups. We already know that they are closed under inverses, and we need only show that they are closed under the group operation.

To that end, assume that $\mathbf{a}_1$ and $\mathbf{a}_2$ both belong to A, but their product $\mathbf{b} = \mathbf{a}_1 * \mathbf{a}_2$ does not belong to A. Then $\mathbf{b}$ is a nonzero element of either B or C, say B. Being perpendicular to $\mathbf{a}_1$, it is also perpendicular to $\mathbf{a}_1^{-1}$, and therefore

$$\mathbf{a}_1^{-1} \times \mathbf{b} = \mathbf{a}_1^{-1} * \mathbf{b} = \mathbf{a}_2\,.$$

But this means that $\mathbf{a}_2$ is perpendicular to $\mathbf{a}_1^{-1}$, which is impossible, since both of them belong to A. Thus A is closed under the group operation and hence is a subgroup (and likewise B and C are subgroups of G). The normality of A follows from the relation

$$\mathbf{b}^{-1} \times (\mathbf{a} \times \mathbf{b}) = -\mathbf{a}\,.$$

(This element has to belong to G since it is $\mathbf{b}^{-1} * \mathbf{a} * \mathbf{b}$. Obviously it lies on the same line through the origin as $\mathbf{a}$ and hence is in A.)

Now suppose all three of A, B, and C have elements of G different from $\mathbf{e}$. Let $\mathbf{b} \in B \setminus \{\mathbf{e}\}$, for example. We claim that the coset of $\mathbf{b}$ relative to A is precisely the set

$$(C \setminus \{\mathbf{e}\}) \cup \{\mathbf{b}\}\,.$$

It is obvious that any element of this coset other than $\mathbf{b}$ itself must belong to C, since $\mathbf{a} * \mathbf{b} = \mathbf{a} \times \mathbf{b}$ is non-zero and perpendicular to both A and B. Conversely, if $\mathbf{c} \in C \setminus \{\mathbf{e}\}$, then

$$\mathbf{c} = \mathbf{a} * \mathbf{b}\,,$$

where

$$\mathbf{a} = -\mathbf{b}^{-1} \times \mathbf{c} = \mathbf{c} \times \mathbf{b}^{-1} \in A\,.$$

(Again, $\mathbf{a}$ belongs to G, since the cross product here can be replaced by the group operation.) Given this characterization of the cosets relative to A, we see that either $C \setminus \{\mathbf{e}\} = \varnothing$, or else B contains only one element different from $\mathbf{e}$. (Distinct cosets have to be disjoint.) The first case is what we hope to conclude. The second case implies the interesting result that G is the Klein four-group.

As remarked, in the first case, it follows by the same argument that $B \setminus \{\mathbf{e}\} = \varnothing = C \setminus \{\mathbf{e}\}$, and we are done. It remains to rule out the case in which G is the Klein four-group. This is easy to do. For, since the cross product of two distinct non-zero elements is non-zero, it must be equal to the group product of those elements. But the Klein four-group is commutative, and the vector cross-product is anticommutative. Hence, $B \setminus \{\mathbf{e}\}$ and $C \setminus \{\mathbf{e}\}$ are empty, and we are done.

Problem A6

Let $f : [0, \infty) \to \mathbb{R}$ be a strictly decreasing continuous function such that $\lim_{x\to\infty} f(x) = 0$.

Prove that $\displaystyle\int_0^\infty \frac{f(x) - f(x+1)}{f(x)}\,dx$ diverges.

Solution: We need a lemma: *Let $a_n > 0$, $n = 0, 1, \ldots$, and suppose the series $\sum_{n=0}^{\infty} a_n$ converges. If $T_n = \sum_{k=n}^{\infty} a_k$, then the series $\sum_{n=0}^{\infty} \frac{a_n}{T_n}$ diverges.*

Proof: Fix any integer m. Then for $n \geq m$ we have

$$\begin{aligned}\frac{a_m}{T_m} + \frac{a_{m+1}}{T_{m+1}} + \cdots + \frac{a_n}{T_n} &\geq \frac{a_m + a_{m+1} + \cdots + a_n}{T_m} \\ &= \frac{a_m + \cdots + a_n}{a_m + \cdots + a_n + T_{n+1}}\,.\end{aligned}$$

But obviously this last fraction tends to 1 as $n \to \infty$, and so the series fails the Cauchy convergence criterion.

That being established, let $a_n(x) = f(x+n) - f(x+n+1)$, so that $T_n(x) = f(x+n)$. Since

$$\sum_{n=0}^{\infty} a_n(x) = f(x)\,,$$

it follows that

$$\sum_{n=0}^{\infty} \frac{a_n(x)}{T_n(x)}$$

diverges for every value of x. By the monotone convergence theorem, this means that

$$\sum_{n=0}^{\infty} \int_0^1 \frac{a_n(x)}{T_n(x)}\,dx = \int_0^{\infty} \frac{f(x)-f(x+1)}{f(x)}\,dx$$

diverges.

Remark. The use of the monotone convergence theorem can be avoided here by considering any positive number M and letting $g_m(x) = \sum_{n=0}^{m} \frac{a_n(x)}{T_n(x)}$. Since $g_m(x)$ increases to infinity, if we set $U_m = \{x \in [0,\infty) : g_m(x) > M\}$, the sets U_m, which are open sets in $[0,\infty)$ because the functions $g_m(x)$ are all continuous, increase with m, and their union is $[0,\infty)$. By the Heine–Borel theorem, there is a finite index m_0 such that $[0,1] \subset U_{m_0}$, that is to say

$$g_m(x) > M$$

for all $x \in [0,1]$ and all $m \geq m_0$. But this means

$$\int_0^m \frac{f(x)-f(x+1)}{f(x)}\,dx = \int_0^1 \sum_{n=0}^{m} \frac{a_n(x)}{T_n(x)}\,dx = \int_0^1 g_m(x)\,dx > M$$

for all $m \geq m_0$. This says precisely that the integral diverges.

Problem B1

Is there an infinite sequence of real numbers a_1, a_2, a_3,... such that

$$a_1^m + a_2^m + a_3^m + \cdots = m$$

for every positive integer m

Solution: No, there is no such sequence. By taking m even we can see that we cannot have $|a_k| > 1$ for any k whatsoever. For $a_k^{2m} > 2m$ if m is sufficiently large and $|a_k| > 1$. But that means $|a_k| \leq 1$ for all k, and hence that

$$\sum_{k=1}^{\infty} a_k^{2m}\,,$$

is a *decreasing* function of m.

Problem B2

Given that A, B, and C are noncollinear points in the plane with integer coordinates such that the distances AB, AC, and BC are integers, what is the smallest possible value of AB?

Solution: The minimal length is 3, obtainable by taking, say, $A = (0, 0)$, $B = (3, 0)$, $C = (3, 4)$ or $C = (10, 24)$.

There is no loss of generality in moving A to $(0, 0)$. If $AB \leq 3$, then B has to be on the same horizontal or vertical line as $(0, 0)$, since there are no other points with integer coordinates at distance 1, 2, or 3 from $(0, 0)$.

If $AB < 3$, we can rule out the possibility that ABC is a right triangle, since the sides of an integer-sided right triangle are $m^2 - n^2$, $2mn$, $m^2 + n^2$ with $1 \leq n < m$. That prevents $m^2 - n^2 = 1$ or $m^2 - n^2 = 2$, since $m^2 - n^2 \geq m + n \geq 3$. Also, $2mn = 2$ implies $m = n = 1$, again an impossibility. Therefore if such a triangle exists with $B = (1, 0)$ or $B = (2, 0)$ we must have (again, without loss of generality), $C = (x, y)$ with $x > 1$ in the first case and $x > 2$ in the second.

Taking the case $B = (1, 0)$ first, we see that

$$AC^2 = x^2 + y^2 \,; \;\; BC^2 = (x - 1)^2 + y^2 \,,$$

and therefore

$$AC^2 - BC^2 = 2x - 1 \,.$$

But $AC \geq BC + 1$ and therefore

$$AC^2 - BC^2 \geq 2BC + 1 > 2x - 1 \,,$$

since $BC > x - 1$.

Similarly, for the case $B = (2, 0)$, we have

$$AC^2 - BC^2 = 4x - 4 \,.$$

This equation means that AC and BC have the same parity and hence differ by at least 2. It follows that $AC \geq BC + 2$, and so

$$AC^2 - BC^2 \geq 4BC + 4 > 4x - 4 \,,$$

again, since $BC > x - 2$.

Problem B3

There are 2010 boxes labeled B_1, B_2,..., B_{2010}, and $2010n$ balls have been distributed among them, for some positive integer n. You may redistribute the balls by a sequence of moves, each of which consists of choosing an i and moving *exactly* i balls from box B_i into any one other box. For which values of n is it possible to reach the distribution with exactly n balls in each box, regardless of the initial distribution of balls?

Solution: This is possible for $n \geq 1005$, but not for $n < 1005$. If the initial distribution is such that box B_i contains fewer than i balls, then no moves can be performed, and it is possible to accommodate

$$1 + 2 + \cdots + 2009 = \frac{2009}{2} 2010 = (1004.5) \times 2010$$

balls in this way. But if $n \geq 1005$, then there will be an excess of balls in some of the boxes, and those can be removed and dumped in box B_1, which will consequently have at least 1005 balls in it after all the boxes are reduced so that there are fewer than i balls in B_i. The balls from the first box can then be used to fill up any non-empty box B_j, $2 \leq j \leq 1006$, until it has j balls in it, which can then be removed and placed in box B_1 again. In this way we get boxes $B_2, \ldots, B_{1006}$ empty, and all of $B_{1007}, \ldots, B_{2010}$ underfilled. But in that case, there are at least

$$2010n - (1006 + \cdots + 2009) = 2010n - 1506 \times 1005$$

balls in B_1. That is more than enough to fill box B_{2010} with $n + 2010$ balls (one at a time out of box B_1), since

$$2010n - (1506)(1005) - \big(n + 2010\big) = 2009n - 1508 \times 1005\,.$$

This is positive, since

$$n \geq 1005 > \frac{1508}{2009} \times 1005\,.$$

We can then remove 2010 of them to B_1, which will then contain at least

$$2009n - (1006 + \cdots + 2008) = 2009n - 2009 \times 1004 + 503 \times 1005$$

balls. Again this is at least $n + 2009$, the difference being

$$2008n - 1506 \times 1005\,,$$

which is positive since

$$n \geq 1005 > \frac{1506}{2008} \times 1005\,.$$

So, we fill B_{2009} with $n + 2009$ balls, then remove 2009 to B_1, which then contains at least

$$2008n - 1004 \times 2007 + 503 \times 1005$$

balls. Again this is at least $n + 2008$, since the difference is

$$2007n - 1504 \times 1005\,,$$

and again

$$n \geq 1005 > \frac{1504}{2007} \times 1005\,.$$

In general,

$$(2010 - n)n - (1510 - 2n) \times 1005$$

is positive, since

$$n \geq 1005 > \frac{1510 - 2n}{2010 - n} \times 1005.$$

Thus, we can keep this up until every box has exactly n balls in it.

Problem B4

Find all pairs of polynomials $p(x)$ and $q(x)$ with real coefficients for which

$$p(x)q(x+1) - p(x+1)q(x) = 1\,.$$

Solution: $p(x) = p_0 + p_1 x$, $q(x) = q_0 + q_1 x$, where $p_0 q_1 - p_1 q_0 = 1$. To prove this, we first show that, except for this case with $p_1 = 0$ or $q_1 = 0$ (but not both, obviously), any polynomials $p(x)$, $q(x)$ satisfying this relationship must have the same degree. Certainly neither of $p(x)$ or $q(x)$

can be constant if the other is of degree higher than 1, since the differences $p(x+1)-p(x)$ and $q(x+1)-q(x)$ have degree only one less than the degrees of $p(x)$ and $q(x)$ respectively.

Now let

$$p(x) = p_0 + p_1 x + \cdots + p_m x^m \quad p_m \neq 0\,, \quad m \geq 1\,,$$
$$q(x) = q_0 + q_1 x + \cdots + q_n x^n\,, \quad q_n \neq 0\,, \quad n \geq 1\,.$$

The degree of $p(x)q(x+1)-p(x+1)q(x)$ is at most $m+n-1$, and the coefficient of x^{m+n-1} in it is

$$(m-n)p_m q_n\,.$$

This must be zero, since $m+n-1>0$. Since $p_m q_n \neq 0$, it follows that $m=n$.

Dividing the equation

$$p(x)q(x+1)-p(x+1)q(x)=1$$

by $p_m q_n$, we see that there are monic polynomials $P(x)=p(x)/p_m$ and $Q(x)=q(x)/q_n$ such that

$$P(x)Q(x+1)-P(x+1)Q(x)=C\,,$$

where C is a non-zero constant. Conversely, if we can find two such monic polynomials $P(x)$ and $Q(x)$, we can divide them by a and b respectively, where $ab=C$, and get polynomials $p(x)=P(x)/a$, $q(x)=Q(x)/b$ satisfying the original equation. Our challenge now is to show that there are no such monic polynomials of degree 2 or higher. (As already shown, they would necessarily be of the same degree.) In fact, we shall show that if $p(x)q(x+1)-p(x+1)q(x)$ is constant, then $p(x)=q(x)$.

To that end, we now assume that

$$p(x) = x^n + p_{n-1}x^{n-1} + \cdots + p_1 x + p_0\,,$$
$$q(x) = x^n + q_{n-1}x^{n-1} + \cdots + q_1 x + q_0\,,$$

$n>1$, and

$$T(x) \equiv p(x)q(x+1)-p(x+1)q(x)=c\,,$$

for some constant c. We mean to show that $c=0$ if $n>1$. We have already noted above that, even without assuming $T(x)$ constant, the degree of $T(x)$ is at most $2n-2$, merely as a consequence of the fact that $p(x)$ and $q(x)$ are polynomials of the same degree. Now in fact, the coefficient of x^{2n-2} in $T(x)$ is $p_{n-1}-q_{n-1}$. Hence, if $T(x)$ is constant and $n>1$, we must have $p_{n-1}=q_{n-1}$. (Note: It is here that we use the assumption that $n>1$. Otherwise, this conclusion does not hold. The relation $p(x)q(x+1)-p(x+1)q(x)=c$, when $p(x)=x+a$ and $q(x)=x+b$ implies only that $a-b=c$, as we have already seen.)

Now suppose that for some nonnegative integer $k \leq n-2$ we have $p_j=q_j$ for $k<j\leq n$. We just showed that this is true for $k=n-2$ if $n>1$.

Then let

$$p(x) = a(x) + p_{k-1}x^{k-1} + p_k x^k + r(x) + x^n\,,$$
$$q(x) = b(x) + q_{k-1}x^{k-1} + q_k x^k + r(x) + x^n\,,$$

where $a(x)$ and $b(x)$ contain only terms of degree less than $k-1$ and $r(x)$ only terms of degree larger than k and less than n. Again, we are assuming that $1 \leq k \leq n-2$. If $k=n-1$, then $r(x)=0$; and if $k=1$, then $a(x)=b(x)=0$.

Then

$$p(x+1) = A(x) + P_{k-1}x^{k-1} + (p_k+R_k)x^k + R(x) + x^n\,,$$
$$q(x+1) = B(x) + Q_{k-1}x^{k-1} + (q_k+R_k)x^k + R(x) + x^n\,,$$

where $A(x)$ and $B(x)$ are of degree less than $k-1$, and $R(x)$ contains only terms of degree larger than k and less than n. The coefficient R_k and the terms in $R(x)$ come from expanding $r(x+1)+(x+1)^n$, and hence are the same for both $p(x+1)$ and $q(x+1)$. For definiteness, let

$$r(x) = r_{k+1}x^{k+1} + \cdots + r_{n-1}x^{n-1}\,,$$
$$R(x) = R_{k+1}x^{k+1} + \cdots + R_{n-1}x^{n-1}\,.$$

We note in particular that $R_{n-1} = r_{n-1} + n$. Then the coefficient of x^{n+k} in the polynomial $p(x)q(x+1)$ is

$$p_k + q_k + R_k + R_{k+1}r_{n-1} + \cdots + R_{n-1}r_{k+1}\,.$$

The coefficient of this same power in $p(x+1)q(x)$ is exactly the same number, and hence this term drops out of $p(x)q(x+1) - p(x+1)q(x)$, which is consequently of degree at most $n+k-1$. The coefficient of x^{n+k-1} in $p(x)q(x+1) - p(x+1)q(x)$ is

$$(p_k - q_k)(R_{n-1} - r_{n-1}) = n(p_k - q_k)\,.$$

Since this must be 0 if $n+k > 1$, it follows that $p_k = q_k$. This completes the induction and shows that $p_k = q_k$ for all k, down to and including $k = 0$, if $n > 1$.

Problem B5

Is there a strictly increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = f\big(f(x)\big)$ for all x?

Solution: No, there is not. Suppose f were such a function. The equation $f'(x) = f\big(f(x)\big)$ shows that $f'(x)$ is also a strictly increasing function, and since it is nonnegative, being the derivative of an increasing function, it must be strictly positive. In particular, if $x > 0$, then $f'(x) > f'(0) = f\big(f(0)\big) > 0$. Likewise

$$f''(x) = f'\big(f(x)\big)f'(x)\,,$$

being a product of two positive, strictly increasing functions, is also positive and strictly increasing. Now fix some value a such that $f(a) = A > 0$, $f'(a) = B > 0$ and $f''(a) = C > 0$. Then for $x > a$ we have

$$f(x) = A + \int_a^x f'(t)\,dt = A + B(x-a) + \int_a^x \int_a^t f''(s)\,ds\,dt \geq A + B(x-a) + \frac{1}{2}C(x-a)^2\,.$$

In particular, $f(x) > x+1$ for all sufficiently large x. Fix a positive value x_0 such that $f(x) > x+1$ for all $x > x_0$. Then for any $x > x_0$, we have

$$\int_x^{f(x)} f'(t)\,dt = f\big(f(x)\big) - f(x) = f'(x) - f(x)\,.$$

On the other hand,

$$\int_x^{f(x)} f'(t)\,dt = \big(f(x) - x\big)f'(c) \text{ for some } c \in [x, f(x)]\,.$$

It follows that

$$f'(x) - f(x) \geq \big(f(x) - x\big)f'(x)\,,$$

which means that

$$0 > -f(x) \geq \big(f(x) - x - 1\big) f'(x) > 0\,.$$

Problem B6

Let A be an $n \times n$ matrix of real numbers for some $n \geq 1$. For each positive integer k, let $A^{[k]}$ be the matrix obtained by raising each entry to the kth power. Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k \geq 1$.

Solution: Fix a row of A, say $(a_{i1}, a_{i2}, \ldots, a_{in})$, and consider the vectors

$$\mathbf{a}_{ir} = (a_{i1}^r, \ldots, a_{in}^r)\,,$$

for $r = 1, 2, \ldots, n$. We claim that the dimension of the vector space spanned by $\{\mathbf{a}_{i1}, \ldots, \mathbf{a}_{in}\}$ equals the number of distinct non-zero entries among $a_{i1}, \ldots, a_{in}$. Suppose without loss of generality that the first k components are all distinct and non-zero, and all other components are equal either to one of these k or to zero. Since

$$\det \begin{pmatrix} a_{i1} & \cdots & a_{ik} \\ a_{i1}^2 & \cdots & a_{ik}^2 \\ \vdots & \vdots & \vdots \\ a_{i1}^k & \cdots & a_{ik}^k \end{pmatrix} = a_{i1} \cdots a_{ik} \big(a_{i1} - a_{i2}\big)(a_{i1} - a_{i3}) \cdots (a_{i\,k-1} - a_{ik}) \neq 0\,,$$

the rows of this matrix span all of $\mathbb{R}^k$. In particular, for any positive integer m we can choose constants $c_{m1}, \ldots, c_{mk}$, $m = 1, \ldots, k$, such that

$$c_{m1}(a_{i1}, \ldots, a_{ik}) + c_{m2}(a_{i1}^2, \ldots, a_{ik}^2) + \cdots + c_{mk}(a_{i1}^k, \ldots, a_{ik}^k) = (a_{i1}^m, \ldots a_{ik}^m)\,.$$

But then, since each other component is equal to one of the first k components or to 0, it follows that

$$c_{m1}\mathbf{a}_{i1} + \cdots c_{mk}\mathbf{a}_{ik} = (a_{i1}^m, \ldots, a_{in}^m)\,.$$

Now of course $k \leq n$. It therefore follows that if a vector $(b_1, \ldots, b_n)$ is orthogonal to $\mathbf{a}_{ir}$ for $r \leq n$, then it is orthogonal to all $\mathbf{a}_{ir}$ for all positive integers r.

Now the hypothesis that $A^{[k]} = A^k$ for $k = 2, \ldots, n+1$ can be phrased by saying that for each fixed i and j, the vector

$$(a_{1j}, a_{2j}, \ldots, a_{j-1\,j}, a_{jj} - a_{ij}, \ldots, a_{nj})$$

is orthogonal to $\mathbf{a}_{ir}$ for $r \leq n$. It is therefore orthogonal to this vector for all r, which is equivalent to saying that the relation $A^{[k]} = A^k$ holds for all k.

Remark: It is clear that every diagonal matrix has this property, as does every matrix in which one row (or one column) has all entries equal and all other rows (or columns) zero. There are still others, for example,

$$\begin{pmatrix} 0 & 0 & 0 \\ b & b & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A complete characterization of such matrices might be interesting.