A Fibonacci-Type Approximation to the Square Root of 2

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We define a sequence of integers $\{a_n\}_{n=1}^{\infty}$ as follows

$$\begin{aligned} a_1 &= 1\,, \\ a_2 &= 1\,, \\ a_{2n+1} &= a_{2n} + a_{2n-1} \quad n = 1\,,\,2\,,\,\ldots\,, \\ a_{2n+2} &= a_{2n} + 2a_{2n-1} \quad n = 1\,,\,2\,,\,\ldots\,. \end{aligned}$$

Thus we get the sequence

$$\{1\,,\,1\,,\,2\,,\,3\,,\,5\,,\,7\,,\,12\,,\,17\,,\,29\,,\,41\,,\,\ldots\}\,.$$

In terms of this sequence, we define a second sequence of rational numbers $\{r_n\}_{n=1}^{\infty}$ as follows:

$$r_n = \frac{a_{2n}}{a_{2n-1}}\,.$$

Thus we have the sequence

$$\left\{1\,,\,\frac{3}{2}\,,\,\frac{7}{5}\,,\,\frac{17}{12}\,,\,\frac{41}{29}\,,\ldots\,\right\}.$$

We claim that

$$\lim_{n\to\infty} r_n = \sqrt{2}\,.$$

To prove this, we observe first that

$$r_{n+1} = \frac{a_{2n+2}}{a_{2n+1}} = \frac{a_{2n} + 2a_{2n-1}}{a_{2n} + a_{2n-1}} = 1 + \frac{a_{2n-1}}{a_{2n} + a_{2n-1}} = 1 + \frac{1}{1 + r_n}\,.$$

Before proceeding to the general proof (which is not long or difficult), we remark that this relation can be rewritten as

$$1 + r_{n+1} = 2 + \frac{1}{1 + r_n}\,,$$

and then, by an easy induction

$$\begin{aligned} r_{n+1} &= 1 + \frac{1}{1 + r_n}\,, \\ &= 1 + \frac{1}{2 + \frac{1}{1 + r_{n-1}}} \\ &= 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{1 + r_{n-2}}}}\,. \end{aligned}$$

Thus we see that if we let n tend to infinity, we would have

$$r_\infty = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{\ddots}}}}}\,.$$

This is well-known to be the continued-fraction expansion of $\sqrt{2}$, and hence we should not be surprised that r_{2n-1} increases to $\sqrt{2}$ and r_{2n} decreases to $\sqrt{2}$.

We can get both of these results at a single stroke, remarking first of all that obviously $r_n \geq 1$ for all n.

Proof of the limit. Since we are interested in making the limit equal to $\sqrt{2}$, it makes most sense to rewrite the recursion equation in terms of $r_n - \sqrt{2}$ and $r_{n+1} - \sqrt{2}$. If we do so, and rationalize the numerator on the right, we get

$$r_{n+1} - \sqrt{2} = \frac{-(r_n - \sqrt{2})}{(1+\sqrt{2})(1+r_n)} \; ;$$

and since $r_n \geq 1$, this implies first of all that $r_{n+1} - \sqrt{2}$ and $r_n - \sqrt{2}$ have opposite signs, that is, the terms fall short of $\sqrt{2}$ or exceed it, according as n is odd or even. Furthermore,

$$|r_{n+1} - \sqrt{2}| \leq \frac{|r_n - \sqrt{2}|}{2 + 2\sqrt{2}} < \frac{|r_n - \sqrt{2}|}{4.8} \, .$$

This means immediately that the sequence $\{r_n\}_{n=1}^{\infty}$ converges to $\sqrt{2}$, and faster than $\left\{\frac{1}{4.8^{n-1}}\right\}_{n=1}^{\infty}$ converges to zero!

Remark: The same limit would result from the recursion relations, no matter what positive numbers were taken as a_1 and a_2. Instead of $\{r_n\}_{n=1}^{\infty}$, the sequence generated would be $\left\{\frac{cr_{n-1}+2}{c+r_{n-1}}\right\}_{n=1}^{\infty}$, where $c = \frac{a_2}{a_1}$ and $r_0 = \infty$. It is trivial that this sequence converges to $\sqrt{2}$ if $r_n \to \sqrt{2}$.

A *Mathematica* Implementation

Here is a simple *Mathematica* program that computes r_n for $n = 1\,,\, 2\,,\, \ldots\,,\, 12$.

```
f[x_] := (2 + x)/(1 + x); rn = Table[Nest[f,1,k-1], {k,1,12}];
b = Table[ N[(4.8^(-1) Abs[rn[[k]] - Sqrt[2]]]/ Abs[rn[[k + 1]] - Sqrt[2]], {k, 1,
11}];
```

The function `rn` gives $r_1\,,\, r_2\,,\, \ldots\,,\, r_{12}$ as the sequence

$$\left\{1\,,\, \frac{3}{2}\,,\, \frac{7}{5}\,,\, \frac{17}{12}\,,\, \frac{41}{29}\,,\, \frac{99}{70}\,,\, \frac{239}{169}\,,\, \frac{577}{408}\,,\, \frac{1393}{985}\,,\, \frac{3363}{2378}\,,\, \frac{8119}{5741}\,,\, \frac{19601}{13860}\right\}.$$

The function `b` gives the first 11 quotients $4.8^{-1}|r_k - \sqrt{2}|/|r_{k+1} - \sqrt{2}|$ as

$$\{1.00592\,,\, 1.2574\,,\, 1.20711\,,\, 1.21549\,,\, 1.21404\,,\, 1.21429\,,\, 1.21425\,,\, 1.21426\,,\, 1.21426\,,\, 1.21426\,,\, 1.21426\}\,.$$

Error computations. Now $r_{12} = \frac{19601}{13860} = 1.41\overline{421356}$, where the overlined six-digit sequence repeats *ad infinitum*. Up to 10 significant figures, $\sqrt{2} = 1.414213562$. As you can see, the error in this approximation is less than 10^{-8}. By our inequality, it should be less than $(\sqrt{2} - r_1) \times 4.8^{-11}$, that is, less than half of 4.8^{-11}, and in fact, $4.8^{-11} = 3.20883 \times 10^{-8}$.